Welfare program of the Jordan basin: formulation, decomposition, and solution through a homotopy algorithm

by

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References
Abstract

The paper specifies a spatially explicit model with a very large number of, say, about 50000 cells, to represent hydrological flows in the Jordan basin within a welfare context. Its main emphasis is on specifying a new homotopy algorithm to solve this model in the presence of market imperfections (missing markets) through which no payment is received for outflows from designated sites of arbitrary spatial configuration (lines, dots, closed objects). Hence, it can represent any coalition of territories within the basin, who refuse to pay to downstream districts as well as imperfections in the monitoring system. It can be run for several years in sequence, as the future can be considered to lie downstream of the present, and in the absence of imperfections will yield an intertemporally efficient path. The equilibrium is unique. The homotopy algorithm is based on price iteration over a sequence of dual convex programs, which are solvable in closed form, on a site-by-site basis, in decreasing order of elevation. We also discuss extensions, including market imperfections, dynamics and the treatment of uninhabited sites.
1. Introduction

The modeling of water flows in a river basin or watershed is the hydrologist’s task. As long as the economic use of the water is primarily dictated by its natural availability and not significantly under control of human activity, hydrological models also offer an adequate tool to represent the creation of economic value from water. Under such conditions, the economic assessment of water flows is confined to the cost-benefit analysis of specified interventions in the basin, such as the construction of a dam, and to the imputation of economic value to the sources of water, such as rainfall, and the evaluation of the contribution of specified territories.

Indeed, where rivers are very large and rainfall is plenty, or where land is sparsely populated, this representation of water flows as a natural process is adequate and the hydrologist’s formulation of the model in terms on non-linear difference equations can be used. In Keyzer (2000, 2002) and Albersen et al. (2002), we indicated how economic valuations could be conducted using such models, without altering them in any way.

However, where water flows have largely been brought under human control, through sluices, pumps and dams, as is the case in the Jordan basin, water use efficiency becomes an issue, and the problem of optimal water allocation has to be addressed. In fact, the distinction between controlled and uncontrolled flows may even arise in a single watershed, with an upstream zone where the flows are natural, a middle zone with significant water control, and towards the estuary a natural section again. The optimization of the model will then have to embrace the territory of the middle zone, that is between the first interventions upstream, and the last interventions downstream, since this is where allocation decisions are made.

The present paper describes the optimization problem for such a middle zone. We formulate this problem as a welfare program that needs to impose far more structure on the modeling of hydrological flows that is the case for other zones.

First, whereas in a purely hydrological model the form of the functional relationships follows from the underlying natural processes, in a welfare program the principles of production theory must be adhered to. Once water flows are subject to human intervention, they may be seen as inputs and outputs of production processes that generally possess the key property of divisibility in the sense that multiple processes can be run at every location, say, cultivating different crops on a given parcel of land. This divisibility makes it possible to circumvent major difficulties in representing the input output relationship of individual crops, and to represent the process as a bundle of given alternatives from which the decision may choose in variable proportions. Clearly, the divisibility property does not apply to major infrastructural works, such as dams or canals, for which the cost-benefit assessment of implementation can be effectuated by comparison of discrete options. The relationships in hydrological models usually do not possess this divisibility property.

Second, in the middle zone geographical detail matters. While it would be possible to treat a zone without human intervention as a large reservoir with a particular pattern of discharges, the optimal management of human intervention calls for representations that are spatially explicit, so as to represent settlement patterns in villages and towns, and the suitability of the land for cultivation. Moreover, a spatially explicit representation greatly eases the task of communication with stakeholders.

Third, once flows come under human control, efficiency of use becomes an issue. In particular, every geographical unit usually has far more control over the volume leaving than the volume entering its territory, as retaining water is easier than attracting it. We focus on the case where the incoming flows are fully beyond the control of the recipient, because they are exclusively gravity driven. Then, to allow for efficiency of allocations it is essential that every
unit be faced with appropriate prices for its downstream deliveries and be rewarded for these, or
on the contrary charged for its water use. In the absence of such payments, the individual
decisions at the various locations will be inappropriately co-ordinated and external effects will
plague the economy, as the interest of downstream users will be neglected.

The main purpose of the model in this paper is to highlight, through a convex, spatially
explicit model with a fine resolution, the welfare gains that can be achieved in the Jordan
watershed, through appropriate co-ordination of decisions, in particular between the Israel,
Jordan, and the Palestinian Authority, with due reward for the upstream neighbors Lebanon and
Syria.

A major difficulty to resolved that such a model is very large, another that it consists of
site-specific models that have to incorporate several technological relations, and, consequently,
include a relatively large number of bounds and constraints. Moreover, because of the nature of
the hydrological process itself, whereby, unlike, say, gasoline, water is never allocated to a single
destination where it can disappear, the site-specific models lack the usual decentralizability
properties of production models.

This raises various challenges for algorithm design, and the aim of the present paper is to
propose suitable restrictions on model specification for addressing these. These restrictions are on
the one hand such that the model decomposes in a sequence of site-specific modules, which
maximize their profit for given delivery prices downstream, and given inflows from sites
upstream, and on the other hand that every site-specific model admits a solution in closed form.
We point out that the resulting wedge between delivery and shadow price can be interpreted as a
tax. The basic principle of the algorithm is to reduce these taxes proportionately for all sites. As
shown in Ginsburgh and Keyzer (2002, ch. 5), such a reduction raises welfare globally, but the
difficulty to be dealt with here is that the choice variables of the algorithm are the delivery prices
rather than the wedges, which are endogenous. Yet, we can prove convergence of the
proportionate tax-reduction procedure by means of a bound-retraction argument, which
establishes that, locally, the proportionate reduction in price wedges is implementable in the sense
that it maintains feasibility of the prevailing solution, reduces actual taxes and eventually leads to
the zero-tax situation that solves the original welfare program.

The paper proceeds as follows. Section 2 discusses some of the main features of water as
an economic good, and their implications for modeling. Section 3 specifies a model for an
individual site in the basin, using assumptions that permit to solve the model in analytical form,
without iteration. Section 4 presents the welfare program and section 5 formulates a dedicated
algorithm to solve it in a large-scale, spatially explicit setting. Section 6 considers various
extensions, including market imperfections, dynamics and the treatment of uninhabited sites.
2. Modeling water as an economic good

2.1 Key features

Water is an economic good because it is scarce. Salt is a pollutant that can be interpreted as using up scarce unpolluted water. Hence, both are commodities in an economic sense, one a good, other a “bad”. As water and salt are highly divisible, their flows and stocks can be represented by real variables, which from an optimization viewpoint eases their welfare maximizing allocation in space and time. However, like any other natural resource (air or land) salinated water has specific economic features that distinguish it from, say, bread.

(1) Time, space, and contiguity Formally, in an economic analysis, all goods that are being distinguished in a model classification are different, as soon as they carry a price of their own. As water can be distinguished by its spatial location at various points in time, every drop can in principle be viewed as a different good, and the water flow as a transformation process in space and in time. Flows only move to contiguous locations in space, and from present to the immediate future in time.

(2) Gravity. Gravity offers the main organizing principle. In fact, the specificity of the treatment of a particular good in an economic model stems from the availability of such a principle, that makes it possible to make the distinctive feature explicit, rather than keeping it hidden in the numerical value of model parameters. Gravity imposes a fixed hierarchical structure on flows in space whereby sites can be visited in sequence, from top to bottom altitude. Thus, gravity imposes a recursiveness in space similar to what causality imposes in time. Furthermore, as water is bulky, its transport by other means than gravity is generally uneconomical. Hence, like land, water is above all an input in primary production and goods produced with water are more likely to be transported by human activity than water itself.

(3) Fluidity. The outflow depends on stock level and the slope. This relationship constitutes the central element of hydrology. From a modeling perspective fluidity is the main distinctive element. It essentially means smoothness of response to inflows. There are two aspects. First, the response functions of outflows to inflows are smooth. Secondly, the response is distributed over time. To represent this, hydrologists model flows as entries into and discharges from reservoirs (bathtubs). A flow will tend to average out over time, as it takes time for the various bathtubs of the hydrological system to fill or be emptied, and downstream effects depend on stocks in the tubs, rather than on incoming flows. Yet, in a steady state, stocks are constant, and outgoing flows depend on exogenous flows entering the system.

Laws of mechanics. The fluid obeys basic laws of mechanics. The law of conservation of mass which implies that all destinations must be accounted for and are governed by specific rules. Furthermore, the functional forms characterizing the processes are given from the natural sciences and not by micro-economic theory of production.

2.2 Implications for modeling

Time-space and contiguity. The model describes a single year, and distinguishes water flows by month. Space is subdivided into S cells or sites, indexed s and r. These could be envisaged to represent points on a raster. The partitioning with a grid of squares, allowing for movement in
eight directions (union jack) is the simplest since it only requires memorizing the boundaries of the territory. However, in an application with a fine resolution, that includes rivers, deserted areas, and intensively cultivated fields, a raster would have to be very fine to preserve the underlying geographical information. Polygons that partition space in relatively homogeneous subsets are better suited to this task. In our model sites refer to polygons that fully partition the space. For this, the sets $N_s^+, N_s^-$ define the collection of upstream and downstream neighbors, respectively. As we seek to present a geographical map, after the welfare program has been solved, it is necessary to convert the sites back to a regular grid i.e. to rasterize them. For this, we can maintain a mapping $s(\tilde{s})$, where $\tilde{s}$ are sites on the regular (fine) grid. Calculations involving, say, the surface of the $s$-“polygon” may be required to infer values at $\tilde{s}$ level from data at the coarser $s$-level.

**Gravity.** Natural flows are only downstream. In addition at specified spots, water pumping can be represented. Gravity allows for spatial recursiveness, and through it for the major simplifying restrictions supporting a decentralized procedure for solving the large-scale model.

**Fluidity.** Suppose that water outflow $y$ was a smooth function of inflow $q$: $y = aq$; this would amount to direct impulse response as a shock in input is immediately translated into a shock in output, without delay, as in an electric wire. This is adequate to represent water flows in closed pipes but neglects the fluidity of surface and groundwater flows that can contract and expand at virtually every spot. To represent the bathtubs the model should allow for stocks: $k = q + k_0 - y$, $y = ak$. Yet, hydrologists often model the flows as the discretization of a differential equation: $y = k = f(k)$. Over a given time period, the outflow is then calculated by approximating the integral $y_t = \int_0^t f(k(\tau))d\tau = F(k_t) - F(k_0)$. The integral operation is to represent the emptying and filling of bathtubs explicitly, allowing for intra-period distribution across date points $t$. It is possible to approximate the differential equation itself by a discrete representation on a monthly basis: $y_{\tau+1} = f(k_\tau)$ while $k_{\tau+1} = k_\tau - y_\tau$, $\tau = 1, 2, ..., 12$. Hence, stock accumulation can be obtained by the iteration until $\tau = 12$, and for positive stocks.

**Laws of mechanics.** Generally, the laws of mechanics imply lack of free disposal, rule out the possibility of inaction and imply that functional forms are to respect specifications from the natural sciences, to the extent possible.

Basically, the main challenge for modeling is to represent the above features within a welfare program that is convex. Without such convexity it is not possible to solve an optimization problems with this large number of choice variables. At the end of our exposition we assess whether the convexity requirement is a major limitation.

Regarding fluidity, the speed of outflow to downstream sites, and of local absorption and release is determined by a variety of circumstances commonly studied by hydrologists (ref.**). In the present paper we initially use a relatively simple pattern, with surface outflow within the month, and a one month delay for exchanges between surface and groundwater, and between the

\[\text{We can use sparse matrix notation to stack this information, } r_s^+ \text{ denoting the location in the stack vector } h^+ \text{ where the site number for upwards origins of } s \text{ is stored (until } r_{s+1}^+-1), \text{ and similarly for downward destinations. To each element is also associated a flow share } \alpha_{rs} \text{ (stored directly or evaluated via a function) which measures the fraction of the water originating from } r \text{ that flows to } s.\]
groundwater layers. In section 6, we show that more flexibly distributed patterns can be accommodated as well.

2.3 Representation within a convex welfare program

Before we present the detailed spatial equilibrium developed for application to the basin, we formulate a stylized version that highlights its major features. For this, we start from the model of the individual site that meets the stated requirements with respect to time-space dynamics, gravity and fluidity. Next, we construct the associated spatial equilibrium model. Finally, we show that, under given assumptions, this model possesses a unique equilibrium solution which appears to be a welfare optimum.

Stylized model of the site

In the stylized model, we distinguish a single time period, and two layers of land at every site: a surface layer (denoted by subscript \(i = 1\)), and subsurface layer (\(i = 2\)) that coincides with the rooting zone of crops. A given site is faced with given precipitation \(b_i\) and lateral inflow \(q_i\) into the surface layer, which originates from more elevated sites (gravity requirement).

The model is developed around the water balance for both layers. The water balance for the surface layer assigns the initial stock plus lateral inflow \(k_i^0\) from upstream sites plus precipitation \(b_i\), plus the upward flow from the sub-surface layer \(h_i\), to availability for use \(a_i\) and to infiltration \(s_i\) into the sub-surface layer:

\[
a_i + s_i = h_i + b_i + q_i + k_i^0
\]  

(2.1)

while in the sub-surface layer the initial stock plus the infiltration from the surface layer is assigned to then the upward flow \(h_i\) and the end-stock

\[
h_i + k_i^2 = s_i + k_i^0.
\]  

(2.2)

The absorption \(s_i\) of surface water depends on the gross water availability \(\hat{a}_i\):

\[
S_i = \sigma_i(\hat{a}_i),
\]  

(2.3)

for \(\hat{a}_i = b_i + q_i + k_i^0\), where \(\sigma_i\) is convex increasing and homogeneous with slope less than one. However, the subsurface layer reaches saturation at a maximal end-stock level of \(\bar{k}_2\):

\[
k_i^2 = \min(\hat{a}_2, \bar{k}_2),
\]  

(2.4)

for gross availability \(\hat{a}_2 = s_i + k_i^0\). Hence, the net absorption is \(k_i^2 - k_i^0 = S_i - h_i\) for

\[
h_i = \max(\hat{a}_2 - \bar{k}_2, 0).
\]  

(2.5)
Economic activity (crop farming $c$) causes evaporation and thus reduces the water volume available for outflow; the available surface water $a_l$ is used either for cultivation $c$ or for non-use $n_l$. Both yield an outflow and an end-stock:

\[
y_l^c = y_l^c(c_l) + y_l^n(n_l)
\]
\[
k_l^c = k_l^c(c_l) + k_l^n(n_l)
\]
\[
c_l + n_l = a_l
\]

where all flow and stock functions are concave increasing, homogeneous and differentiable. Moreover, we assume that for all $c_l$ and $n_l$:

\[
y_l^c(c_l) + y_l^n(n_l) + k_l^c(c_l) + k_l^n(n_l) + \ell_1 = c_l + n_l,
\]

allows for a nonnegative loss $\ell_1$ through evaporation.

Cultivation generates a revenue or production function $f(c_l)$, which is concave increasing in $c$. Since the output prices are taken as given, we can think of $f$ as being expressed in monetary terms and interchangeably refer to revenue and production. End-stocks are valued at positive stock prices $\psi_1$ and $\psi_2$ for surface and subsurface flows, respectively, while the downstream runoff is valued at the positive price $\pi_1$. In-site availability is taken to be valued at a given opportunity cost $\pi_1$. Maximizing this objective subject to constraints (2.1)-(2.6), we obtain the site-specific program:

\[
V(q_1, k^2_l) = \max_{a_l, c_l, q_l, k^1_l, n_l, y_l} f(c_l) + \pi_1 y_l + \psi_1 k^1_l + \psi_2 k^2_l
\]

subject to

\[
k^1_l = k^c_l(c_l) + k^n_l(n_l)
\]
\[
y_l = y^c_l(c_l) + y^n_l(n_l)
\]
\[
c_l + n_l = a_l
\]
\[
a_l = b_l + q_l + k^0_l - (k^1_l - k^0_l)
\]

for given

\[
k^2_l = \min(\sigma_l(b_l + q_l + k^0_l) + k^0_l, k^2_l).
\]

Now $V$ is decreasing in $k^1_l$. Since the slope of $\sigma_l$ w.r.t. $q_l$ is less than unity, $a_l(q_l, k^1_l(q_l))$ is increasing in $q_l$, both for $q_l > \hat{q}_l$ and for $q_l < \hat{q}_l$, for $\hat{q}_l$ such that $\sigma_l(b_l + q_l + k^0_l) + k^0_l = \bar{k}_2$. Moreover, $V(q_l, k^1_l(q_l))$ is concave increasing in $q_l$ in both situations. However, the slope of $a_l(q_l, k^1_l(q_l))$ is suddenly rising to unity at $q_l = \hat{q}_l$, causing a non-convexity in the program. This non-convexity is basic, as it reflects an increasing returns to scale feature, whereby a site is better able to transfer and use its water when the subsurface reservoirs have been replenished. Hence, a water allocation authority might choose to exercise concentration in its channeling of water to economic use. Yet, to keep the numerical procedure tractable we limit our attention to a version without non-convexity. For this, we introduce the simplification that the outcome of the
maximum in (2.8) is known a priori. This simplification may not be too severe when we distinguish flows by month (i.e. take (2.7)-(2.8) to apply to a single month), and keep the area of individual sites sufficiently large. Thus, we replace (2.8) by:

$$k^2_i = \delta (\sigma_i(b_i + q_i + k^0_i) + k^0_2) + (1 - \delta) \bar{k}_2,$$

(2.9)

where $\delta$ is 0 or 1, depending on the site and the month.

**Spatial equilibrium**

The spatial equilibrium model is formulated by distinguishing $S$ sites indexed $s$, in the basin or watershed, with a site-specific model (2.7)-(2.8) for every site

$$V_s(b_{1s}, q_{1s}, k^0_{1s}, k^0_{2s}, \pi_{1s}, \psi_{1s}, \psi_{2s}) = \max_{a_{1s}, c_{1s}, k^l_{1s}, n_{1s}, y_{1s}, y_{l1s} \geq 0} g_s(c_{1s}) + \pi_{1s} y_{1s} + \psi_{1s} k^l_{1s} + \psi_{2s} k^0_{2s}$$

subject to

$$k^l_{1s} = k^c_{1s}(c_{1s}) + k^0_{1s}(n_{1s})$$
$$y_{1s} = y^c_{1s}(c_{1s}) + y^0_{1s}(n_{1s})$$
$$c_{1s} + n_{1s} = a_{1s}$$
$$a_{1s} = b_{1s} + q_{1s} + k^0_{1s} - (k^l_{2s} - k^0_{2s})$$

for given

$$k^2_{2s} = \delta (\sigma_s(b_{1s} + q_{1s} + k^0_{1s}) + k^0_2) + (1 - \delta) \bar{k}_2.$$  

(2.11)

We want to solve this program as well as to evaluate its marginal value of availability $p_{1s} = \nabla b_{1s} V$, which measures the marginal contribution to the objective of an increase in surface water availability.

Gravity offers the main organizing principle, since it permits to address the sites in sequence, starting from the highest elevation:

$$q_{1s} = \frac{1}{(1 + p_{1r})} y_{lr} \alpha_{rs},$$

(2.12)

where $\alpha_{rs}$ is the share of the flow from site $r$ that accrues to site $s$. Clearly, the share is non-negative and sums to unity. Finally, to co-ordinate the decisions at the various sites optimally, we seek selling prices to downstream locations that satisfy the price equilibrium condition:

$$\pi_{1s} = \frac{1}{(1 + p_{1s})} \sum_r \alpha_{sr} p_{lr}.$$  

(2.13)

**Welfare program**

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2 Appendix B provides an algorithm to demarcate a watershed on a map.
Model (2.10)-(2.13) defines the first-order conditions of the welfare program, that is we determine the flows that maximize the value of output and final stocks, subject to the constraints:

\[
W = \max_{a_{1s}, c_{1s}, k_{1s}^l, k_{2s}^l, n_{1s}, q_{1s}, y_{1s}} \sum s \left( f_s(c_{1s}) + \psi_{1s}k_{1s}^l + \psi_{2s}k_{2s}^l \right) \\
\text{subject to} \\
k_{1s}^l = k_{1s}^l(c_{1s}) + k_{1s}^n(n_{1s}) \\
y_{1s} = y_{1s}^l(c_{1s}) + y_{1s}^n(n_{1s}) \\
c_{1s} + n_{1s} = a_{1s} \\
a_{1s} = b_{1s} + q_{1s} + k_{1s}^0 - (k_{2s}^l - k_{2s}^0) \\
k_{2s}^l = \delta_s(\sigma_{1s}(b_{1s} + q_{1s} + k_{1s}^0) + k_{2s}^0) + (1 - \delta_s)k_{2s}^r \\
q_{1s} = \sum_r \frac{1}{(1 + \rho_{lr})} y_{1r} \alpha_{rs},
\]

whose feasibility depends on the nonnegativity of surface water availability \(a_{1s}\). Solving this welfare program permits to check whether the assumptions on \(\delta_s\) holds. Convexity of this program ensures that a stationary point of (2.14) is a global welfare optimum. Hence, a spatial equilibrium (2.10)-(2.13) is a welfare optimum.
3. Further elaboration of the site-specific model

The present section considers several extensions of the site-specific model: from two period to twelve months per cycle; a surface, a subsurface and a groundwater layer; flows and stocks of salt and water. We subsequently restrict the formulation because we need a solution in closed form to obtain an algorithm that can efficiently find a spatial equilibrium.

3.1 Monthly production function

Production is treated as a land use process whose intensity depends on the water stock that is available in every month, as well as on direct precipitation $b$, and inflows from other sites $q$. Water can be used in production of, say, a crop, or passed on to the downstream sites or cells via a stream and surface runoff $y$. Use in production implies in part evaporation and in part accumulation of stock of soil moisture, and groundwater. Salt is diluted in water, and obviously, does not evaporate.

A water input pattern $c_{\tau}$ in months $\tau = 1, \ldots, T$, for $T = 12$ leads to an annual production equal to $\min_{\tau} g_{\tau}(c_{\tau})$. This production structure possesses the following properties.

**Assumption P (production):** (a) The production function has an output expressed in monetary units discounted to the beginning of the year and obeys $\min_{\tau} g_{\tau}(c_{\tau})$, where $c_{\tau}$ is the water input in month $\tau$; (b) the monthly production functions satisfy $g_{\tau}(c_{\tau}) = f(c_{\tau} + \eta_{\tau})/k_{\tau}$, where $k_{\tau}$ is positive and $\eta_{\tau}$ unconstrained in sign, and $f$ is homogeneous, strictly concave increasing and has a Lipschitz continuous derivative, and is also well defined for negative $c$; (c) for given positive constants $f_{0}$ and $f'_{0}$, both $f(c) = f_{0}$ and $\frac{\partial f(c)}{\partial c} = f'_{0}$ possess a solution in analytical form whenever a solution exists.

We note that months with zero marginal demand for water $k_{\tau} = 0$, say, during the post-harvest period, can be dealt with separately, but for convenience we take this coefficient to be strictly positive in every month.

3.2 Three layers

We distinguish between surface water, subsurface water reaching until the root zone of crops, and groundwater. For this, we introduce the index $i$, $i = 1,2,3$. Model (2.10)-(2.13) can be rewritten:
\[
V = \max_{a_{i\tau}, c_{i\tau}, k_{i\tau}, n_{i\tau}, y_{i\tau}, y_{i\tau} \geq 0, c} f(c) + \sum_{i} \psi_{i} k_{i\tau} + \sum_{i} \sum_{\tau} \tau y_{i\tau}
\]
subject to
\[
c_{i\tau} \geq \eta_{c} c - \eta_{c}^{0}
\]
\[
k_{i\tau} = k_{i\tau}^{c}(c_{i\tau}) + k_{i\tau}^{n}(n_{i\tau})
\]
\[
y_{i\tau} = y_{i\tau}^{c}(c_{i\tau}) + y_{i\tau}^{n}(n_{i\tau})
\]
\[
c_{i\tau} + n_{i\tau} = a_{i\tau}
\]
\[
a_{i\tau} = b_{i\tau} + q_{i\tau} + k_{i,\tau-1}
\]
\[
+ \delta_{\tau}(\nu_{2\tau}(k_{2,\tau-1}) - \sigma_{i\tau}(b_{i\tau} + q_{i\tau} + k_{i,\tau-1}))
\]
\[
+ (1 - \delta_{\tau})(k_{2,\tau-1} + \nu_{2\tau}(k_{3,\tau-1}) - \sigma_{i\tau}(k_{2,\tau-1}) - \bar{k}_{2\tau})
\]
\[
k_{2\tau} = \delta_{\tau}[\sigma_{i\tau}(b_{i\tau} + q_{i\tau} + k_{i,\tau-1}) - \sigma_{2\tau}(k_{2,\tau-1})]
\]
\[
+ \nu_{3\tau}(k_{3,\tau-1}) - \nu_{3\tau}(k_{2,\tau-1}) + k_{2,\tau-1}] + (1 - \delta_{\tau})k_{2\tau}
\]
\[
k_{3\tau} = \sigma_{2\tau}(k_{2,\tau-1}) - \nu_{3\tau}(k_{3,\tau-1}) + k_{3,\tau-1}.
\]

given \(k_{10}\), while \(\tau = 1, ..., T\) and \(T = 12\), and where the percolation, absorption or seepage functions \(\sigma_{2\tau}\) and \(\sigma_{3\tau}\) are linear and homogeneous, and the same holds for the “capillarity” functions \(\nu_{3\tau}\) (\(\nu\) for upward flow that describe natural upward flows. The price \(\psi_{i}\) now refers to month 12, i.e. the end-stock in the last month of the year. The reason to keep these functions linear is that this maintains convexity of the constraint set: the functions appear with in the equation for sub-surface stock and the equation for groundwater stock with an opposite sign.

We observe that, through the inequality constraint, and the fact that the scalar \(c\) is unconstrained in sign while inflows are non-negative, we can always achieve a feasible allocation, with \(c_{i\tau} = 0\). This allocation may, however, yield negative income. Non-negative income can be ensured by allowing for choice between landuse types. We return to this aspect in section 5.3 below.

### 3.3 Representing salinity; piecewise linearization

Next, salt is accounted for in each of these zones. We focus on salinization because in the Jordan basin this is the most direct threat to crop production. Salt is disposed of with the harvesting of crops that absorb it, and, more importantly, with runoff to downstream sites. In high densities salt kills all life and the Dead Sea where the Jordan River ends, owes its name and reputation to its high degree of salinity.

To represent this, we maintain a record of salt accumulation in each layer. Thus, we modify the model presented so far. We distinguish commodities indexed \(i = 1, 2,..., m, m = 6\). We also distinguish between commodities that contribute to production (“goods” or water: \(I^{+} = \{1, 2, 3\}\), for surface, sub-surface and groundwater), and pollutants (“bads” or salt \(I^{-} = \{4, 5, 6\}\) for surface, sub-surface and groundwater) that frustrate production.

The distinctive properties to be accounted for are that (a) salt does not evaporate; (b) it has a negative impact on crop production; and (c) it only leaves the site with the water flows and is distributed in the same shares as water to the downstream sites. Property (a) means that for salt...
the commodity balances hold with strict equality, without any loss. To incorporate property (b), we include an additional step in the mapping between $c$ and $c_{l\tau}$:

$$r_{l\tau} = \eta^c_{l\tau}c + \eta^0_{l\tau}$$

$$c_{l\tau} = r_{l\tau} + \eta^0_{l\tau}a_{4\tau}.$$  (3.2a, 3.2b)

**Assumption WR (water requirement functions):** the water requirement functions $r_{l\tau}(c)$ are convex increasing, homogeneous and piecewise linear: $c_{l\tau} = \eta^c_{l\tau}c + \eta^0_{l\tau} + \eta^0_{l\tau}a_{4\tau}$; the function has fixed threshold points $\hat{c}_\ell$, $\ell = 1,\ldots,L$ on the line segment $\hat{c}_{\ell-1} \leq c \leq \hat{c}_\ell$;

$\eta^c_{\ell-1\tau}\hat{c}_\ell + \eta^0_{\ell-1\tau} = \eta^c_{\ell\tau}\hat{c}_\ell + \eta^0_{\ell\tau}c_0 << 0$; $\eta^0_{l\tau} = 0$; $0 \leq \eta^0_{\ell-1\tau} \leq \eta^0_{\ell\tau}$: $\eta^0_{\ell\tau} \leq \eta^0_{\ell\tau}$.

The total water requirement $c_{l\tau}$ of the crop is used for three purposes: evaporation, flushing and storage in the sub-surface. A higher yield requires more flushing, among others to dispose of the salt. We return to this aspect below. Hence, associated to the monthly water requirements are the flushing and the storage functions. In the sequel we denote all piecewise linear function in the same way as the water requirement, with a regime dependent slope and intercept.

**Assumption WOS (water outflow and storage):** The water outflow and the storage functions $y^c_{l\tau}(c_{l\tau})$, $k^c_{l\tau}(c_{l\tau})$, $y^n_{l\tau}(n_{l\tau})$ and $k^n_{l\tau}(n_{l\tau})$ are (i) piecewise linear, homogeneous, convex, with fixed switch points $c_{l\tau}(\hat{c}_\ell)$; (ii) $y^c_{l\tau}(c_{l\tau}) \leq y^n_{l\tau}(c_{l\tau})$ and $k^c_{l\tau}(c_{l\tau}) \leq k^n_{l\tau}(c_{l\tau})$ (iii) they satisfy $y^c_{l\tau}(c_{l\tau}) + k^c_{l\tau}(c_{l\tau}) \leq c_{l\tau}$, and $y^n_{l\tau}(n_{l\tau}) + k^n_{l\tau}(n_{l\tau}) \leq n_{l\tau}$, for all $c_{l\tau}$ and $n_{l\tau} \in [0, \bar{a}]$.

We use the symbols $\kappa$ and $\zeta$ for the coefficients of these $k$- and $y$-functions, respectively. To incorporate property (c), we formulate outflow functions:

**Assumption SO (salt outflow):** The salt outflow functions $y^c_{4\tau}(c_{1\tau})$ and $y^n_{4\tau}(c_{1\tau})$, are piecewise linear, homogeneous and convex in $c_{1\tau}$, with fixed switch points $c_{1\tau}(\hat{c}_\ell)$.

We use the symbol $\mu$ for the coefficients of these $y$-functions. Finally, we postulate the simplest possible specification for the downward and upward streams between layers.

**Assumption WSC (water and salt: seepage and capillary flows):** The seepage and capillary flow functions $\sigma$ and $\nu$, are fixed, regime independent, but site and month-specific fractions.

**Assumption SD (salt damage):** In the salt damage function $\eta_y^s$ is a fixed, nonnegative coefficient.

Now substituting the piecewise linear relationships, the site-specific program becomes:
\[ V = \max_{a_{1,\tau}, c_{1,\tau}, k_{1,\tau}, n_{1,\tau}, y_{4,\tau}, y_{6,\tau} \geq 0, c} \, f(c) + \sum_{i=1}^{\tau} \psi_i k_{i,\tau} + \sum_{i=\tau}^{\tau} \pi_i y_{i,\tau} \]

subject to

water constraints:

\[ c_{1,\tau} = \eta_{c,1,\tau} c + \eta^o_{c,1,\tau} + \eta^a_{c,1,\tau} \]
\[ k_{1,\tau} = \eta^c_{c,1,\tau} c + \eta^c_{k,1,\tau} n_{1,\tau} + \kappa^0_{c,1,\tau} \]
\[ y_{1,\tau} = \xi_{c,1,\tau} c + \xi^c_{k,1,\tau} n_{1,\tau} + \zeta_{c,1,\tau} \]
\[ c_{1,\tau} + n_{1,\tau} = a_{1,\tau} \]
\[ a_{1,\tau} = b_{1,\tau} + q_{1,\tau} + k_{1,\tau-1} \]
\[ + \delta_{c,1,\tau}(\psi_{2,\tau} k_{2,\tau-1} - \sigma_{1,\tau}(b_{1,\tau} + q_{1,\tau} + k_{1,\tau-1})) \]
\[ + (1 - \delta_{c,1,\tau})(\psi_{3,\tau} k_{3,\tau-1} - \sigma_{2,\tau} k_{2,\tau-1} - \bar{k}_{2,\tau}) \]
\[ k_{2,\tau} = \delta_{c,1,\tau}([\sigma_{1,\tau}(b_{1,\tau} + q_{1,\tau} + k_{1,\tau-1}) - \sigma_{2,\tau} k_{2,\tau-1} \]
\[ + \psi_{3,\tau} k_{3,\tau-1} - \psi_{2,\tau} k_{2,\tau-1} + k_{2,\tau-1}] + (1 - \delta_{c,1,\tau})\bar{k}_{2,\tau} \]
\[ k_{3,\tau} = \sigma_{2,\tau} k_{2,\tau-1} + (1 - \psi_{3,\tau}) k_{3,\tau-1}, \]

salt constraints:

\[ y_{4,\tau} = \mu^c_{c,1,\tau} c + \mu^c_{k,1,\tau} n_{1,\tau} + \mu^o_{c,1,\tau} \]
\[ k_{4,\tau} = a_{4,\tau} - y_{4,\tau} + \psi_{5,\tau} k_{5,\tau-1} - \sigma_{4,\tau} a_{4,\tau} \]
\[ k_{5,\tau} = k_{5,\tau-1} + \psi_{6,\tau} k_{6,\tau-1} - \sigma_{5,\tau} k_{5,\tau-1} - \psi_{5,\tau} k_{5,\tau-1} + \sigma_{4,\tau} a_{4,\tau} \]
\[ a_{4,\tau} = b_{4,\tau} + q_{4,\tau} + k_{4,\tau-1} \]
\[ k_{6,\tau} = k_{6,\tau-1} - \psi_{6,\tau} k_{6,\tau-1} + \sigma_{5,\tau} k_{5,\tau-1}. \]

Here we combine the constants out the outflow functions into a single coefficient, with superscript \( o \). These salt constraints satisfy the commodity balance and can for given \( c_{i,\tau} \) and \( n_{i,\tau} \) be solved moving from the bottom-layer constraint upwards.

Remark: Representation of tap-water

Most of the tap water in Israel is pumped from Lake Tiberias, into the “National Carrier”, a pipeline system that provides most of the country with tap water. In modeling terms such a system differs from regular surface flows in that demand determines extraction at every site, while the details of the sewage system determine whether the water is re-cycled or disposed of in open streams. In the simplest representation, the waterworks can be looked at as a single site, with outflows to many other sites depending on the sewage technology.
4. Closed form solution of the site-specific model with piecewise linear constraints

4.1 The solution form

In this section, we seek a closed form solution of the model. The first step is to substitute the water requirement into the water outflow and storage functions, and the salt outflow functions. Then, under regime \( \ell \), the constraints of (3.4) consist of the bounds:

\[
a_{I\tau}, a_{4\tau}, c_{I\tau}, n_{I\tau}, y_{I\tau}, y_{4\tau} \geq 0, \ i_1 \geq 0
\]

and the equalities:

**water constraints:**

\[
k_{1\tau} = (\kappa_{I\tau}^c - \kappa_{I\tau}^a)(\eta_{I\tau}^c c + \eta_{I\tau}^a a_{I\tau}) + \kappa_{I\tau}^n a_{I\tau} + \kappa_{I\tau}^0 \tag{4.1}
\]

\[
y_{I\tau} = (\zeta_{I\tau}^c - \zeta_{I\tau}^a)(\eta_{I\tau}^c c + \eta_{I\tau}^a a_{I\tau}) + \zeta_{I\tau}^n a_{I\tau} + \zeta_{I\tau}^0
\]

\[
a_{I\tau} = b_{I\tau} + q_{I\tau} + k_{I\tau-l}
\]

\[
+ \delta_{I}\left(\nu_{2\tau} k_{2\tau-l} - \sigma_{I\tau}(b_{I\tau} + q_{I\tau} + k_{I\tau-l})\right)
\]

\[
+ (1 - \delta_{I})(k_{2\tau-l} + \nu_{3\tau} k_{3\tau-l} - \sigma_{I\tau}^2 k_{2\tau-l} - k_{2\tau})
\]

\[
k_{2\tau} = \delta_{I}\left[\sigma_{I\tau}(b_{I\tau} + q_{I\tau} + k_{I\tau-l}) - \sigma_{I\tau}^2 k_{2\tau-l}
\]

\[
+ \nu_{3\tau} k_{3\tau-l} - \nu_{2\tau} k_{2\tau-l} + k_{2\tau-l}\right] + (1 - \delta_{I})k_{2\tau}
\]

\[
k_{3\tau} = \sigma_{I\tau} k_{2\tau-l} + (1 - \nu_{3\tau}) k_{3\tau-l}
\]

**salt constraints:**

\[
y_{4\tau} = (\mu_{I\tau}^c - \mu_{I\tau}^a)(\eta_{I\tau}^c c + \eta_{I\tau}^a a_{I\tau}) + \mu_{I\tau}^n a_{I\tau} + \mu_{I\tau}^0
\]

\[
k_{4\tau} = a_{4\tau} - y_{4\tau} + \nu_{5\tau} k_{5\tau-l} - \sigma_{I\tau} a_{4\tau}
\]

\[
k_{5\tau} = k_{5\tau-l} + \nu_{6\tau} k_{6\tau-l} - \sigma_{I\tau} k_{5\tau-l} - \nu_{5\tau} k_{5\tau-l} + \sigma_{I\tau} a_{4\tau}
\]

\[
a_{4\tau} = b_{4\tau} + q_{4\tau} + k_{4\tau-l}
\]

\[
k_{6\tau} = k_{6\tau-l} - \nu_{6\tau} k_{6\tau-l} + \sigma_{I\tau} k_{5\tau-l}
\]

Observe that as long as \( y_{I\tau}, k_{I\tau}, y_{4\tau} \) and \( k_{4\tau} \) are non-negative, all other variables, except \( c \), must be nonnegative, and that for \( c \) negative enough and \( k_{4\tau} \) non-negative, these variables are nonnegative.
4.2 The constraints in matrix form

Next, we substitute out the availability of water and we write in matrix-form the stock accumulation by regime as:

\[ k_{\tau} = T_{\tau} k_{\tau-1} + r_{\tau} + M_{\tau}(q_{\tau} + b_{\tau}) - e_{\tau} c, \]  

(4.2)

for the non-negative, 6×6 transition matrix:

\[
T_{\tau} = \begin{bmatrix}
\kappa_{\tau}^{n}(1-\delta_{\tau}^{n}\sigma_{1_{\tau}}) & \kappa_{\tau}^{n}(\delta_{\tau}^{n}\nu_{2_{\tau}}+(1-\delta_{\tau}^{n})(1-\sigma_{2_{\tau}})) & \kappa_{\tau}^{n}(1-\delta_{\tau}^{n}\nu_{3_{\tau}}) & (\kappa_{\tau}^{n}-\kappa_{\tau}^{n})\eta_{\tau}^{e} & 0 & 0 \\
\delta_{\tau}^{n}\sigma_{1_{\tau}} & \delta_{\tau}^{n}(1-\nu_{2_{\tau}})-\sigma_{2_{\tau}} & \delta_{\tau}^{n}\nu_{3_{\tau}} & 0 & 0 & 0 \\
0 & \sigma_{2_{\tau}} & (1-\nu_{3_{\tau}}) & 0 & 0 & 0 \\
-\mu_{\tau}^{n}(1-\delta_{\tau}^{n}\sigma_{1_{\tau}}) & -\mu_{\tau}^{n}(\delta_{\tau}^{n}\nu_{2_{\tau}}+(1-\delta_{\tau}^{n})(1-\sigma_{2_{\tau}})) & -\mu_{\tau}^{n}(1-\delta_{\tau}^{n}\nu_{3_{\tau}}) & -(1-\sigma_{4_{\tau}})(\mu_{\tau}^{n}-\mu_{\tau}^{n})\eta_{\tau}^{e} & 0 & \sigma_{4_{\tau}}(1-\sigma_{5_{\tau}}) \\
0 & 0 & 0 & 0 & \sigma_{4_{\tau}} & (1-\sigma_{5_{\tau}}) \\
0 & 0 & 0 & 0 & 0 & \sigma_{5_{\tau}}(1-\nu_{6_{\tau}})
\end{bmatrix},
\]

and the 6-dimensional vectors

\[
r_{\tau} = \begin{bmatrix}
-\kappa_{\tau}^{n}(1-\delta_{\tau})\bar{\kappa}_{2_{\tau}}-(\kappa_{\tau}^{n}-\kappa_{\tau}^{n})\eta_{1_{\tau}}^{o}+\kappa_{\tau}^{o} \\
(1-\delta_{\tau})\bar{k}_{2_{\tau}} \\
\mu_{\tau}^{n}(1-\delta_{\tau})\bar{k}_{2_{\tau}}+(\mu_{\tau}^{n}-\mu_{\tau}^{n})\eta_{1_{\tau}}^{o}+\kappa_{\tau}^{o} \\
\end{bmatrix},
\]

\[
e_{\tau} = \begin{bmatrix}
(\kappa_{\tau}^{n}-\kappa_{\tau}^{n})\eta_{\tau}^{e} \\
0 \\
0 \\
-(\mu_{\tau}^{n}-\mu_{\tau}^{n})\eta_{\tau}^{e}
\end{bmatrix},
\]

In solution form this leads to the vector equation:

\[ k_{\tau} = \beta_{\tau} - \gamma_{\tau} c, \]  

(4.3)

where,
\( \gamma_{/\tau} = e_{/\tau} \) for \( \tau = 1 \),

and

\( \gamma_{/\tau} = e_{/\tau} + T_{/\tau} \gamma_{/\tau-1} \) for \( \tau = 2, \ldots, 12 \),

while

\[
\beta_{/\tau} = T_{/\tau} \beta_{/\tau-1} + M_{/\tau}(q_{/\tau} + b_{/\tau}) + r_{/\tau},
\]

for \( \tau = 1, 2, \ldots, 12 \), and \( \beta_{/0} = k_0 \) for \( \tau = 0 \). Moreover, defining the vectors

\[
h^T_{/\tau} = [(1 - \delta_{/\tau} \sigma_{1/\tau}) \delta_{/\tau} v_{2/\tau} + (1 - \delta_{/\tau})(1 - \sigma_{2/\tau}) (1 - \delta_{/\tau}) v_{3/\tau} \ 0 \ 0 \ 0],
\]

and the scalars

\[
m^0_{/\tau} = -(1 - \delta_{/\tau}) \overline{k}_{2/\tau}, \ m^q_{/\tau} = (1 - \delta_{/\tau} \sigma_{1/\tau}),
\]

we obtain for surface water availability the equation

\[
a_{/\tau} = h^T_{/\tau} \beta_{/\tau-1} + m^0_{/\tau} + m^q_{/\tau}(q_{/\tau} + b_{/\tau})
\]

and, therefore, in solution form the water availability is:

\[
a_{/\tau} = h^T_{/\tau} \beta_{/\tau-1} + m^0_{/\tau} + m^q_{/\tau}(q_{/\tau} + b_{/\tau}) - h_{1/\tau} \gamma_{/\tau-1} c,
\]

and the nonnegativity requirement on availability also defines the upper bound \( \overline{c} \).

Finally, substituting the availability into the outflow functions, the outflow of water and salt can be written in solution form as:

\[
y_{1/\tau} = \omega^c_{1/\tau} c + \omega^o_{1/\tau}
\]

for

\[
\omega^c_{1/\tau} = (\zeta^c_{1/\tau} - \zeta^n_{1/\tau}) \eta^c_{1/\tau} - \zeta^n_{1/\tau} \gamma_{1/\tau-1}, \n\]

\[
\omega^o_{1/\tau} = -(\zeta^c_{1/\tau} - \zeta^n_{1/\tau}) \eta^o_{1/\tau} + \zeta^n_{1/\tau}(h^T_{/\tau} \beta_{/\tau-1} + m^0_{/\tau} + m^q_{/\tau}(q_{/\tau} + b_{/\tau})) + \zeta^o_{/\tau},
\]

and

\[
y_{4/\tau} = \omega^c_{4/\tau} c + \omega^o_{4/\tau},
\]

for

\[
\omega^c_{4/\tau} = (\mu^c_{4/\tau} - \mu^n_{4/\tau}) \eta^c_{4/\tau} - \mu^n_{4/\tau} \gamma_{4/\tau-1}, \n\]

\[
\omega^o_{4/\tau} = -(\mu^c_{4/\tau} - \mu^n_{4/\tau}) \eta^o_{4/\tau} + \mu^n_{4/\tau}(h^T_{/\tau} \beta_{/\tau-1} + m^0_{/\tau} + m^q_{/\tau}(q_{/\tau} + b_{/\tau})) + \mu^o_{/\tau}.
\]

Note that \( \omega^o \) is of indefinite sign, and has negative elements for high salinity, while \( \omega^c \) is nonpositive.

### 4.3 Solution in closed form, for two regimes

For given upper bound \( \overline{c} \), which depends on deliveries \( d = q + b \) to the site, we can now write the program in the simple form

\[
y_{1/\tau} = \omega^c_{1/\tau} c + \omega^o_{1/\tau}
\]

for

\[
\omega^c_{1/\tau} = (\zeta^c_{1/\tau} - \zeta^n_{1/\tau}) \eta^c_{1/\tau} - \zeta^n_{1/\tau} \gamma_{1/\tau-1}, \n\]

\[
\omega^o_{1/\tau} = -(\zeta^c_{1/\tau} - \zeta^n_{1/\tau}) \eta^o_{1/\tau} + \zeta^n_{1/\tau}(h^T_{/\tau} \beta_{/\tau-1} + m^0_{/\tau} + m^q_{/\tau}(q_{/\tau} + b_{/\tau})) + \zeta^o_{/\tau},
\]
\[
\max_{c \leq \hat{c}(q+b)} f(c) + \sum_{i} \sum_{\tau} \pi_{i\tau} y_{1\tau}(c) + \sum_{i} \psi_{i\tau} k_{i\tau}(c).
\]  

(4.7)

Before solving this program in closed form, we must deal with the piecewise linearity of Assumption WR, and we must also determine \( \hat{c}_{\tau} \), to enforce the constraint \( c_{1\tau} \leq a_{1\tau} \). For this, we use the non-negativity requirement on \( y_{1\tau}, k_{1\tau}, \) and \( y_{4\tau} \), disregarding the bound on \( k_{4\tau} \), because for this variable the zero stock level only is a benchmark:

\[
\begin{align*}
y_{1\tau} &= \omega_{1\tau}^{c} c + \omega_{1\tau}^{o}, \\
y_{4\tau} &= \omega_{4\tau}^{c} c + \omega_{4\tau}^{o}, \\
k_{1\tau} &= \beta_{1\tau} - \gamma_{1\tau} c
\end{align*}
\]  

(4.8)

This defines the upper bound on \( c \) as:

\[
\hat{c}_{\tau}(q+b) = \min\{\min(-\frac{\omega_{1\tau}^{o}}{\omega_{1\tau}^{c}}, -\frac{\omega_{4\tau}^{o}}{\omega_{4\tau}^{c}}, \beta_{1\tau} / \gamma_{1\tau})\}.
\]  

(4.9)

Since this bound could be negative, we must ensure that the function \( f \) is well defined on the negative axis. For this, we specify it as:

\[
f(c) = f^{o}(\max(c,0)) + \nabla f^{o}(0) \cdot \min(c,0) - \kappa (\min(c,0))^2,
\]  

(4.10)

where the first term is a common production function that is well defined for non-negative \( c \); recall that it has bounded slope \( \nabla f^{o} \); the second term maintains continuity of slope at the origin, and the third term penalizes negative \( c \). We now allow for piecewise linearity in two regimes \( \ell = 1,2 \) only, and distinguish four cases:

<table>
<thead>
<tr>
<th>Upper bound is:</th>
<th>If ( \hat{c}_{1} \geq \hat{c} )</th>
<th>If ( \hat{c}_{1} &lt; \hat{c} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( \hat{c}_{2} &gt; \hat{c} )</td>
<td>( \hat{c} = \hat{c}_{2} )</td>
<td>n.a.</td>
</tr>
<tr>
<td>If ( \hat{c}_{2} \leq \hat{c} )</td>
<td>n.a.</td>
<td>( \hat{c} = \hat{c}_{1} )</td>
</tr>
</tbody>
</table>

It appears that two cases cannot apply; this is because the piecewise linearity and the convexity ensure that \( \hat{c}_{1} > \hat{c}_{2} \) if \( \hat{c}_{2} > \hat{c} \) and that \( \hat{c}_{2} < \hat{c}_{1} \) if \( \hat{c}_{1} < \hat{c} \).

<table>
<thead>
<tr>
<th>Range is:</th>
<th>If ( \hat{c}_{1} \geq \hat{c} )</th>
<th>If ( \hat{c}_{1} &lt; \hat{c} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( \hat{c}_{2} &gt; \hat{c} )</td>
<td>( [c, \hat{c}] \cup [\hat{c}, \hat{c}_{2}] )</td>
<td>n.a.</td>
</tr>
<tr>
<td>If ( \hat{c}_{2} \leq \hat{c} )</td>
<td>n.a.</td>
<td>( [c, \hat{c}_{1}] )</td>
</tr>
</tbody>
</table>

Here the lower bound obeys \( \underline{c} = \min(0, \hat{c}_{1}, \hat{c}_{2}) \). We also assume that the outflow functions are piecewise linear, with derivative \( \mu_{i\tau}^{\ell} \) w.r.t. \( c \). The calculations for solving the site-specific model proceed as follows. For ease of notation, we represent the regime specific values of coefficients by the superscript \( \ell \), and write the algorithm as Fortran-like pseudo code.

Procedure for solving site-specific program:
(1) Recurrently determine for every month $\tau$ and regime $\ell$, the (site-specific) coefficients $\beta_{\ell,\tau}$ and $\gamma_{\ell,\tau}$, moving forward over months, as in (4.4)-(4.6).

(2) Determine the upper bounds $\hat{c}_{\ell}$, as in (4.8).

(3) Evaluate objective coefficients of (4.7):

$$\chi_{\ell} = \sum_{i=1,4}(\psi_{iT}\gamma_{i/T}) - \sum_{\tau}(\pi_{i\tau}\omega_{i\tau}), \quad (4.9)$$

where we note that $\chi_1 \leq \chi_2$.

(4) If solution is to be found on either of the two intervals $[\underline{c},\hat{c}], [\hat{c},\check{c}]$:

If $f'(\hat{c}) > \chi_1$, then

if $f'(\check{c}) > \chi_1$ then
    $c = \check{c}$
else
    $c$ solves $f'(c) = \chi_1$
endif
else
    if $(f'(c_1) < \chi_1)$ then
        $c = c_1$
    else
        $c$ solves $f'(c) = \chi_1$
endif
endif

(5) Alternatively, a solution is to be found on $[\underline{c},\check{c}]$:

If $f'(\check{c}) > \chi_1$, then

$c = \check{c}$
else
    if $(f'(c) < \chi_1)$ then
        $c = c_1$
    else
        $c$ solves $f'(c) = \chi_1$
endif
endif

4.4 Marginal valuation

In the next section, when the various sites are linked, it appears that the water selling prices $\pi_{i,\tau}$ of every site should in the welfare optimum agree with the derivative of $V$ with respect to inflows $q_{i,\tau}$, in the sense that the selling prices are the weighted sum of the marginal contributions to the various client cells. Hence, we also seek an analytical solution for the derivative. The same applies to the derivative with respect to the inflow of salt.
We return to program (4.7), now explicitly indicating the dependence on inflows \( q \). We drop the subscripts for regime and land use type in the understanding that the calculation is only for the line-segments \( \ell \) that are optimal on a given site, for given prices:

\[
V(\pi,d) = \max_{0 \leq c \leq \tilde{c}(d)} f(c) + \sum_i (\sum_r \pi_{ir} \gamma_{ir}(c,d)) + \psi_{it} k_{iT}(c,d)).
\] 

(4.11)

There are now two utilization regimes. For an optimum with \( c^* < \tilde{c}(d) \) the derivative w.r.t. availability is:

\[
\frac{\partial V}{\partial d_{\tau'}} = \sum_i \psi_i \frac{\partial k_{iT}}{\partial d_{\tau'}} + \sum_i \sum_r \pi_{ir} \frac{\partial \gamma_{ir}}{\partial d_{\tau'}},
\]

where non-use determines the marginal value. Alternatively, if \( c^* = \tilde{c}(q) \), we have an additional term, as economic use in this case determines marginal value:

\[
\frac{\partial V}{\partial d_{\tau'}} = \sum_i \psi_i \frac{\partial k_{iT}}{\partial d_{\tau'}} + \sum_i \sum_r \pi_{ir} \frac{\partial \gamma_{ir}}{\partial d_{\tau'}} + (f'(c) + \sum_i \psi_i \frac{\partial k_{iT}}{\partial c} + \sum_i \sum_r \pi_{ir} \frac{\partial \gamma_{ir}}{\partial c}) \frac{\partial \tilde{c}(q)}{\partial q_{\tau'}} \frac{\partial q_{\tau'}}{\partial d_{\tau'}}.
\]

(4.12)

(4.13)

where \( \frac{\partial q_{\tau'}}{\partial d_{\tau'}} \) is equal to unity. These regimes reflect a discontinuity in the derivative, i.e. nondifferentiability; in fact, the choice of line-segment also has such an effect. To quantify these derivatives, we evaluate recurrently, the derivative

\[
\frac{\partial \beta_{\tau}}{\partial d_{\tau'}} = \begin{cases} 
T_{\tau} \frac{\partial \beta_{\tau-1}}{\partial d_{\tau'}} & \text{if } \tau' < \tau \\
M_{\tau} & \text{if } \tau' = \tau, \\
0 & \text{if } \tau' > \tau
\end{cases}
\]

(4.14)

enabling us to obtain

\[
\frac{\partial \gamma_{\tau}}{\partial d_{\tau'}} = \begin{cases} 
\gamma_{\tau}^{\mu} (h_{\tau}^{\mu} \frac{\partial \beta_{\tau-1}}{\partial d_{\tau'}}) & \text{if } \tau' < \tau \\
m_{\tau}^{d_{\tau'}} & \text{if } \tau' = \tau, \\
0 & \text{if } \tau' > \tau
\end{cases}
\]

(4.15a)

where \( m_{\tau}^{d_{\tau'}} = 0 \) whenever \( i \neq 1 \).
\[
\frac{\partial y_{1\tau}}{\partial d_{\tau'}} = \begin{cases} 
\mu^\prime \left( h'_\tau \frac{\partial \beta_{\tau-1}}{\partial d_{\tau'}} \right) & \text{if } \tau' < \tau \\
0 & \text{if } \tau' \geq \tau
\end{cases}
\] (4.15b)

Furthermore,
\[
\frac{\partial y_{1\tau}}{\partial c} = \omega_{1\tau}, \quad \frac{\partial y_{4\tau}}{\partial c} = \omega_{4\tau},
\] (4.16)

and
\[
\frac{\partial k_T}{\partial d_{\tau'}} = \frac{\partial \beta_T}{\partial d_{\tau'}}, \quad \frac{\partial k_T}{\partial c} = -\gamma_T,
\] (4.17)

and, for the upper bound, the derivatives are:
\[
\frac{\partial \widetilde{c}}{\partial q_{\tau'}} = \begin{cases} 
\frac{1}{\gamma_{1,\tau-1}} \left( \frac{h'_{\tau}}{\partial \beta_{\tau-1}} \frac{\partial \beta_{\tau-1}}{\partial d_{\tau'}} \right) & \text{if } \tau' < \tau \\
\frac{1}{\gamma_{1,\tau-1}} m_{\tau} & \text{if } \tau' = \tau \\
0 & \text{otherwise}
\end{cases}
\] (4.18)

where \( \tau \) is the (known) month that the bound is binding \( (a_{1\tau} = c_{1\tau}) \), on the basis of the equality in (4.8). In the next section, we use these derivatives as target values for flow and prices of the sites located just upstream, and adjust the prevailing prices in that direction, possibly allowing for a rerouting of flows to more profitable downstream sites.
5. Solving the welfare program for the watershed

5.1 Model formulation

Section 4 has developed the site-specific model and describes a procedure to determine in closed form, for given selling and end-stock prices, and for given inflows into the site the land use type, the water use, the water outflow, as well as the shadow price of water inflows. We are now ready to specify the spatial welfare program for the watershed.

First, we repeat the key relationship that distributes water and salt flows from \( r \) to \( s \),

\[
q_{its} = \sum_r \frac{1}{(1 + \rho_{itr})} y_{itr} \alpha_{rs},
\]

according to fixed nonnegative fractions \( \alpha_{rs} \).

**Assumption A (adjacency and gravity):** For given site of origin \( r \), the fraction \( \alpha_{rs} \) accruing to destination \( s \) will be zero whenever the destination is non-adjacent to \( r \), and also if it has a higher elevation.

It is important to note that these fractions are the same for all commodities \( i \), reflecting that salt and other commodities that flow with the water are diluted in it. Other commodities do not flow out of the site. If prices \( \pi \) of the site-specific program (3.4) are such that markets for water are competitive (all take prices as given) and these markets clear, then in the absence of further externalities, these prices can be obtained from the welfare program:

\[
W^* = \max_{c_s, d_{its}, q_{its}, y_{its} \geq 0} \sum_s \left( f_s(c_s) + \sum_i \psi_i s k_{its}(c_s, d_s) \right)
\]

subject to

\[
c_s \leq \tilde{c}_s(d_s)
\]

\[
y_{its} = \tilde{y}_{its}(c_s, d_s)
\]

\[
d_{its} = d_{itr} + b_{its}
\]

\[
q_{its} = \sum_r \frac{1}{(1 + \rho_{itr})} y_{itr} \alpha_{rs},
\]

This is a convex program, whose Lagrange multipliers can be obtained via the marginal valuation (4.11), (4.12), which as we recall may exhibit discontinuous switches when bounds become active and regime changes take place.

5.2 Tax-ridden welfare program

Next, we define a tax-ridden welfare program that provides the basis for the algorithm and its convergence properties, supposing throughout that our standard assumptions apply. Taxes on deliveries to the site are supported by subsidies on outflows from it, and are denoted by \( (\xi, \zeta) \), respectively; both can be negative.

The economic interpretation of these extensions with a tax could be that government at every site runs a procurement system, whereby deliveries \( d \) are purchased at fixed price \( \bar{p} \) from
any supplier, while users pay the prevailing market price, and water outflows are taxed in conformity. This enables us to treat the selling prices selling prices $\pi(\tilde{p})$ as given irrespective of the as yet unknown market prices downstream. For a given delivery, the site-specific markets clear at site-specific prices $\tilde{p}$ for these deliveries $d$, and the price wedge between procurement and delivery can be interpreted as a tax. The resulting program for the watershed reads:

$$W(\xi, \zeta) = \max_{0 \leq c_s \leq \xi_s(d_s), \gamma_{its} \geq 0, d_{its}} \sum_s \left( f_s(c_s) + \sum_i \psi_{its} k_{its}(c_s, d_s) + \sum_i \sum_r \xi_{its} d_{its} - \sum_i \sum_r \xi_{its} y_{its} \right)$$

subject to

$$c_s \leq \xi_s(d_s)$$

$$\gamma_{its} = \tilde{\gamma}_{its}(c_s, d_s)$$

$$d_{its} = q_{its} + b_{its}$$

$$q_{its} = \frac{1}{(1 + \rho_{its})} y_{its} \alpha_{rs}.$$  

If we were able to solve this program for every $(\xi, \zeta)$, it would be easy to obtain the undistorted welfare optimum by eliminating the taxes. However, in closed form we can only solve it for the endogenously generated tax wedges $\xi_{its} = \hat{p}_{its} - \tilde{p}_{its}$ and $\zeta_{its} = \pi_{its}(\hat{p}) - \pi_{its}(\tilde{p})$, where $\pi_{its}(p) = \frac{1}{1 + \rho_{its}} \sum_r p_{its} \alpha_{rs}$. We must develop a dedicated algorithm for that case. To simplify the notation, we define for every site the payoff or money-metric utility function:

$$U_s(y_s, d_s) = \max_{0 \leq c_s \leq \xi_s(d_s)} f_s(c_s) + \psi_s^T k_s(c_s, d_s)$$

subject to

$$\tilde{y}_s(c_s, d_s) = y_s.$$  

This program is not necessarily feasible for all $(y_s, d_s)$ values, but we are only concerned with feasible points, and at those points, value function $U_s$ can be shown to be concave in $(-y_s, d_s)$. Program (5.3) now simplifies to:

$$W(\xi, \zeta) = \max_{q_{its}, \gamma_{its} \geq 0, d_{its}} \sum_s U_s(y_s, d_s) + \sum_i \sum_r \xi_{its} d_{its} - \sum_i \sum_r \xi_{its} y_{its}$$

subject to

$$d_{its} = q_{its} + b_{its}$$

$$q_{its} = \frac{1}{(1 + \rho_{its})} y_{its} \alpha_{rs}. $$

We initially treat $U_s$ as being differentiable, but we do not need this property in the proof of convergence. Its first-order conditions include:

$$\frac{\partial U_s}{\partial d_{its}} + \frac{\partial \xi_{its}}{\partial d_{its}} = \hat{p}_s$$

(5.6a)
\[
\frac{\partial U_s}{\partial y_{its}} - \varepsilon_{its} + \pi_{its}(p) \leq 0 \quad \perp y_{its} \geq 0 ,
\]

and w.r.t. to inflows imply:

\[
p_{its} \leq \hat{p}_{its} \quad \perp q_{its} \geq 0 .
\]

Conditions (5.6b), (5.6b) indicate that we may solve the site-specific programs from top to bottom, at given selling prices \(\pi(\hat{p})\), in the way described in the previous section:

\[
V_s(\hat{p},d) = \max_{y_{its} \geq 0} U_s(y_s,d_s) + \sum_l \sum_{\tau} \pi_{its}(\hat{p}) y_{its} ,
\]

while we obtain for prices that they are a subgradient belonging to the subdifferential of \(V_s\):

\[
\hat{p}_{its} \in \partial_{d_{its}} V_s ,
\]

as defined in (4.11)-(4.12) while \(p_{its} = \hat{p}_{its}\) whenever \(q_{its} > 0\), and can, as we have seen, be solved in closed form, and will be negative for salt. Programs (5.4) and (5.7) illustrate that within sites the economic decision problem is far less decentralizable for water than for the goods of a regular production problem, essentially because an allocation of water is non-exclusive: water used in agricultural production also, for a part, flows out of the site for downstream use. Hence, unlike, say, gasoline, water does not pose the choice between using it and selling it for downstream use: water outflow is the byproduct of the site-specific process and not the direct choice variable of the consumer or producer. Consequently, the pricing of water (and salt) should depend on the marginal value of the inflow into similarly interconnected processes at adjacent sites downstream, inclusive of the reward for outflows of these sites. Thus, rewards for downstream use can also permeate to non-adjacent upstream sites.

It remains to establish the direction of change in \(\hat{p}\) in which the distortion must be reduced, so as to solve (5.2). The natural approach will be to shift this parameter towards \(\hat{p}\), and this indeed proves to be an effective procedure. Yet, to establish convergence a formal argument is required, to which we now turn.

### 5.3 Tax-reducing algorithm

To adjust \(\hat{p}\), we propose the strategy of slowly reducing all taxes proportionately, as in Ginsburgh and Keizer (2002, propositions 1.11, 5.3, 5.5), where it is shown that undistorted welfare \(V\), the sum of non-tax terms in (5.5), never drops under such a reduction. The property holds globally and irrespective of the sign of the distortions. Since the tax might also be interpreted as referring to monopoly rents, the idea is more generally to implement a reform agenda of improved competition and economic liberalization.

However, since we can only specify \(\hat{p}\), we must prove that, locally, a change in \(\hat{p}\) made so as to reduce all taxes proportionately for given \(\hat{p}\), will, also after \(\hat{p}\) adjusting to this change, never cause welfare to fall. To establish this, we present two representations that support the same solution as (5.5), at given values of \(\hat{p}'\) for iteration \(t\), \(t = 1,2,\ldots\).

First, we write the actual allocation as a quantity constrained program:
\[
U(q_s', d_s', y_s', z_s') = \max_{q_{irs} \geq 0, d_{irs}, y_{irs}} \sum_s U_s(y_s, d_s)
\]
subject to
\[
\begin{align*}
  d_{irs} &= q_{irs} + b_{irs} \\
  q_{irs} &= \sum_r \frac{1}{1 + \rho_{irs}} y_{irs} \alpha_{rs} . \\
  d_{irs}^l &\leq d_{irs} \leq d_{irs}^u \\
  y_{irs}^l &\leq y_{irs} \leq y_{irs}^u ,
\end{align*}
\]
that relates to the given \( \hat{p}' \) according to:
\[
\hat{p}' = p' - (\xi^{+l} - \xi^{-l})
\]
while the tax-multipliers obey:
\[
\begin{align*}
  \xi^{+l} &= \max(\xi^l, 0), \xi^{-l} = \max(-\xi^l, 0), \xi^{+d} = \max(\zeta^l, 0), \xi^{-d} = \max(-\zeta^l, 0) , \\
  \zeta^l &= \pi(\hat{p}') - \pi(p') ,
\end{align*}
\]
and the bounds are set according to:
\[
\begin{align*}
  d_{ks}^l &= d_{ks} \text{ if } \xi_{ks}^{+l} > 0 \text{ and } d_{ks}^u = d_{ks} + \kappa \text{ otherwise} , \\
  d_{ks}^l &= d_{ks} \text{ if } \xi_{ks}^{+d} > 0 \text{ and } d_{ks}^u = d_{ks} - \kappa \text{ otherwise} , \\
  y_{ks}^l &= y_{ks} \text{ if } \xi_{ks}^{+l} > 0 \text{ and } y_{ks}^u = y_{ks} + \kappa \text{ otherwise} , \\
  y_{ks}^l &= y_{ks} \text{ if } \xi_{ks}^{+d} > 0 \text{ and } y_{ks}^u = y_{ks} + \kappa \text{ otherwise} .
\end{align*}
\]
We are now ready to specify the price adjustment algorithm for \( \hat{p}' \) according to the simple averaging process:
\[
\hat{p}'^{i+1} = (1 - \sigma) \hat{p}'^i + \sigma \hat{p}' .
\]
for a positive stepsize constant \( \sigma \). We observe that this process can be interpreted as a discretization of the differential equations \( \frac{dp}{dt} = \hat{p}(\hat{\theta}, \hat{\theta}) - \hat{p} \). It remains to prove convergence.

**Proposition 1 (Co-operative equilibrium: welfare optimum):** Let the assumptions A, WR, WOS, SO, WSC and SD hold. Then, for positive \( \sigma \) small enough and less than unity, homotopy process (5.13) for \( t = 0, 1, \ldots \), converges to the optimum of welfare program (5.2).

**Proof.** The proof consists of two parts. One is to establish that the sequence of solutions (5.9)-(5.13) is well defined in that the sequence of site specific problems, jointly with the price adjustment rule (5.13) actually implements it. The other is to establish convergence itself.
Part 1. The implementability of the procedures for solving the sequence of site-specific programs was already established in section 4. Specifically, under assumptions $A$, $WR$, $WOS$, $SO$, $WSC$ and $SD$ this generates bounded sequences of (possibly set-valued) prices $\hat{p}^t$, for given $\hat{p}$. Next, to verify that these prices support (5.9)-(5.12), it suffices to compare the first-order conditions, which can be seen to coincide. Hence, the algorithm describes a sequence of programs (5.9) with bounds (5.12).

Part 2. Turning to convergence, we remark that the sequence will be welfare improving if the update of bounds in (5.12) always causes, as compared to the previous iteration, a loosening of some active bound in (5.9), and no tightening of any. In (5.5) imposition of the tax-wedge adjustment to accommodate the change in $\hat{p}$ amounts to a weakening of the penalization on all the effective bounds, since all tax-wedges both $\xi$ and $\zeta$ are being scaled by a common factor $(1 - \sigma)$, in view of the linearity and homogeneity of the selling price function $\pi$, which ensures that $\zeta$ is linear and homogeneous in $\xi$.

Formally, for a program $\max_{(x_1, x_2) \in X} u(x_1, x_2) \mid x_1 \leq \bar{x}_1, x_2 \leq x_2$, where $u$ is strictly concave and $X$ is compact convex, and where all bounds are taken to be effective, the Lagrangean $u^*(\alpha, p_1, p_2) = \max_{(x_1, x_2) \in X} au(x_1, x_2) - \phi_1(x_1 - \bar{x}_1) - \phi_2(x_2 - x_2)$, can be used to analyze the effect of such a proportionate adjustment. Note that a price change by a positive factor $1/\alpha$ has the same effect as a multiplication of utility by the inverse of that factor. Then, from the Envelope theorem it follows that

$$\frac{\partial x_1}{\partial \alpha} = \frac{\partial^2 u^*(\alpha, \phi_1, \phi_2)}{\partial \phi_1 \partial \alpha} = -\frac{\partial u(x(\alpha, \phi_1, \phi_2))}{\partial \phi_1} > 0,$$

$$\frac{\partial x_2}{\partial \alpha} = \frac{\partial^2 u^*(\alpha, \phi_1, \phi_2)}{\partial \phi_2 \partial \alpha} = \frac{\partial u(x(\alpha, \phi_1, \phi_2))}{\partial \phi_2} < 0,$$

establishing the monotonicity: a small rise in $\alpha$ amounts to a proportionate fall by a factor $1/\alpha$ in prices and will cause every bound to be exceeded (note that the proportionality plays a critical role). Hence, locally in (5.5), i.e. for $\sigma$ small enough, the change in $\hat{p}$ will cause $d, y$ to exceed all their active bounds, and since in (5.12) these bounds adjust to the new level, the change is welfare improving in (5.9). As welfare is bounded, the series $U^t, t = 0, 1, \ldots$ is a Cauchy sequence and, therefore, converges to a point $U^*$, where no bound is binding, i.e. where all taxes have been eliminated. It is important to note that the argument circumvents the requirement of single-valuedness of the price mapping $\hat{p}(\hat{p})$, which in view of the regimes and bounds would not be fulfilled, and only requires single-valuedness of this mapping at the point of convergence.

5.4 Multiple landuse types and the transition among them

So far, we only allowed for a single landuse type. A natural generalization is to allow for several types competing at a single location.. We represent these by indexing the site-specific model with an index $j$, and letting the site choose the most rewarding types. The model’s transparency would be served if we could ensure that only one type is selected. However, this amounts to discrete choice, and is not a tractable option from computational perspective.
Before proceeding to the specification of a smoothing mechanism, we note that the socially optimal choice of land-type should be based on surplus rather than revenue comparison, as the effective use of input \( d \) should be accounted for. Hence the optimal discrete choice is to select the landtype on the basis of its surplus:

\[
\Pi_{sj}(\tilde{p}, \tilde{d}_{sj}) = \max_{d_{irsj} \geq 0} V_{sj}(\tilde{p}, \tilde{d}_{sj}) - \sum_{i} \sum_{\tau} \tilde{p}_{irsj} d_{irsj}, \tag{5.14a}
\]

for the modification of site-specific model (5.7)

\[
V_{sj}(\tilde{p}, d_{sj}) = \max_{y_{irsj} \geq 0} U_{sj}(y_{sj}, d_{sj}) + \sum_{i} \sum_{\tau} \pi_{irsj}(\tilde{p}) y_{irsj}. \tag{5.14b}
\]

The surplus is obtained by (4.7), (4.8) from the site-specific revenue, which satisfies:

\[
V_{sj}(\tilde{p}, d_{sj}) = \min_{\tilde{p}_{irsj}} \left( \sum_{i} \sum_{\tau} \tilde{p}_{irsj} d_{irsj} + \Pi_{sj}(\tilde{p}, \tilde{p}_{sj}) \right), \tag{5.15}
\]

i.e. for Lagrange multipliers \( \tilde{p} \) in (4.8), as the surplus function:

\[
\Pi_{sj}(\tilde{p}, \tilde{p}_{sj}) = V_{sj}(\tilde{p}, d_{sj}) - \sum_{i} \sum_{\tau} \tilde{p}_{irsj} d_{irsj}, \tag{5.16}
\]

which, by the Maximum Theorem is a continuous function, despite the set-valued nature of \( \tilde{p} \) in (4.8). In the absence of further restrictions, the selection of optimal landuse type, would be on the basis of maximal surplus

\[
\Pi_{s*}(\tilde{p}, d_{sj}) = \max_{\tilde{p}} \Pi_{sj}(\tilde{p}, \tilde{p}_{sj}(\tilde{p}, d_{sj})) = \max_{w_{sj} \geq 0} \left\{ \sum_{j} w_{sj} \Pi_{sj}(\tilde{p}, \tilde{p}_{sj}(\tilde{p}, d_{sj})) \right\} \sum_{j} w_{sj} = 1 \}. \tag{5.17}
\]

whose weights \( w_{sj} \) coincide with those that are optimal in:

\[
V_{s}(\tilde{p}, d_{sj}) = \min_{\tilde{p}} \sum_{i} \sum_{\tau} \tilde{p}_{irsj} d_{irsj} + \max_{w_{sj} \geq 0} \left\{ \sum_{j} w_{sj} \Pi_{sj}(\tilde{p}, \tilde{p}_{sj}(\tilde{p}, d_{sj})) \right\} \sum_{j} w_{sj} = 1 \}. \tag{5.18}
\]

Now the site-specific value-function \( V_{s} \) is well behaved, since it is continuous, convex in \( \tilde{p} \) and concave in \( d_{sj} \). However, it does not possess the single-valuedness required for Lemma 1. To maintain single valuedness we introduce a portfolio cost function.

**Assumption PC (portfolio cost):** The portfolio cost function \( C_{s} : \mathbb{R}_{+}^{d} \rightarrow \mathbb{R}_{+}, C_{s}(w_{s1}, ..., w_{sj}) \) is (i) homogeneous of degree one; (ii) continuously differentiable; (iii) strictly quasiconvex increasing.

With this cost function, problem (5.17) is modified to:

\[
\Pi_{s*}(\tilde{p}, d_{sj}) = \max_{w_{sj} \geq 0} \left\{ \sum_{j} w_{sj} \Pi_{sj}(\tilde{p}, \tilde{p}_{sj}(\tilde{p}, d_{sj})) - C_{s}(w_{s1}, ..., w_{sj}) \right\} \sum_{j} w_{sj} = 1 \}. \tag{5.19}
\]

and the original payoff function (5.4) becomes:
subject to
\[
\sum_j y_{sj}(c_{sj}, w_{sj}, d_s) = y_s
\]
\[
\sum_j w_{sj} = 1
\]

The weights \(w_{sj}\) now appear as “fixed factors” in all the functions. They can be inserted in these functions, so as to make them homogeneous of degree one. For example, under mild limit conditions at \(w_{sj} = 0\), (see Ginsburgh and Keyzer, 2002, Theorem A.1.5), the “extended” function
\[
f_{sj}(c_{sj}, w_{sj}) = w_{sj} \tilde{f}_{sj}(c_{sj} / w_{sj})
\]
can fulfill this role. For (piecewise) linear functions, it suffices to multiply the intercepts by \(w_{sj}\). Thus, all functions of model (3.4) can be made homogeneous of degree one.

The strict quasiconvexity of the cost function will now ensure single-valuedness of the primal variables. It remains to find an closed form solution. For this, we specify a CES convex function of CES-“plus” type:
\[
\tau_j = \sum_j G(w_{1},...,w_{J}) = \kappa(\sum_j (w_j)^2)^{\frac{1}{2}}.
\]

**Proposition 2 (Portfolio selection: \(\ell_1 + \ell_2\) case):** The portfolio problem (5.20), with cost function (5.21) is solvable exactly, in at most \(J\) iterations. It only selects landuse types with positive revenue.

**Proof.** The algorithm is presented in Proposition A.2 of the Appendix.

This result deserves some further comment. First, the exact solution algorithm can accommodate fairly large numbers of candidate landuse types because its first round already eliminates all those that have negative gross surplus (before deducting the Euclidean parts of the costs). Therefore, one may consider abstaining from any piecewise linearization within the closed form specification of section 4, while maintaining flexibility through the diversity of landuse processes. In the extreme case a landuse type reduces to a single activity vector transforming the given inflow pattern into economic value, on the basis of the availability in the most constraining month.

Second, because commodity balances must hold with equality, the model specified so far could not guarantee that surpluses remain non-negative. This can now be ensured by allowing for some “inactive” landuse types of which at least one has a revenue that is insensitive to salinity.

Third, the interpretation of combination a landuse types at a single site may be problematic, since the data cannot distinguish between landuse types, and the convexity presupposes that all convex combinations of variables are feasible, which may conflict with process knowledge. The limitation is not a serious one, since the cost function can be specified so as to ensure that in the optimum almost every site will operate a single landuse type. This can be implemented by using a small value for the indirect cost constant \(\kappa\) in (5.21) relative to direct costs \(\tau_j\), possibly even reducing it once an optimum has been found to “refine” the solution further.

Finally, with respect to the interpretation of the landuse type we remark that we could use in (5.14b) a \(j\)-specific selling price \(\pi_{ij}(\tilde{p})\). This illustrates that the landuse types may differ in...
their delivery pattern, i.e. that the shares $\alpha_r$ and the loss factors $\rho_s$ may vary among landuse type.
6. Extensions and modifications

6.1 Market imperfections

The co-operative solution of the welfare program is an idealized outcome, that should be contrasted with more realistic situations. In particular we should account for an institutional setting in which sites may not have to pay for the water they receive. Recall, that the selling prices for water and salt obey the relation:

\[ \pi_{tss} = \frac{1}{(1 + \rho_{tss})} \sum_r \alpha_{sr} p_{itr}, \quad i = 1, 4 \]  

(6.1)

and are positive for water and negative for salt. For other commodities, the prices are given. Program (5.2) leads to a welfare optimum that is for a solution of the perfectly co-operative scenario. External effects across sites are introduced by distinguishing within the set of downstream neighbors \( N_s^- \) between a subset \( M_s^- \) containing those who pay for their water use and receive compensation for the salt that accrues to them, and its complement \( N_s^- - M_s^- \). Thus, an asymmetry emerges between the flows (5.1) and their pricing:

\[ \pi_{tss} = \frac{1}{(1 + \rho_{tss})} \sum_{r \in M_s^-} \alpha_{sr} p_{itr}, \quad i = 1, 4. \]  

(6.2)

which can readily be inserted in the site-specific program (5.7). We distinguish between three scenarios: free riding, co-operation, and co-operation within designated territories only. The cooperation scenario was already discussed extensively. Since multiple landuse types could, via Proposition 2, be incorporated in the site-specific model without affecting basic properties we do not distinguish these in the sequel, for ease of exposition.

Free riding

If no one pays the water supplying sites upstream for their deliveries, then for given prices of terminal stocks, the model solution becomes very simple.

**Proposition 3 (Free riding):** Let assumptions A, WR, WOS, SO and SC hold, and suppose that all sites only value their own use \((\pi = 0)\). Then, a single round that runs Procedure 1 for all sites in top-down order is sufficient to solve the welfare program with individualistic agents. ◊

Proof. The site-specific welfare program (5.7) is feasible, because the bounds \( \tilde{c} \) maintain nonnegativity of \( q_{i,s} \) for water, which in turn ensures that the interval \([0, \tilde{c}_{j,s}(q_s)]\) is nonempty. The concavity properties ensure that its stationary point is a global optimum. □

Equilibrium concepts

Next, we turn to cases in which externalities only occur between but not necessarily within national territories.
Surplus maximization (5.14), is the behavioral rule for all sites. A Nash equilibrium is found, when for the sites solved recursively from top to bottom, prices satisfy \( \hat{p} = \hat{p} \). Since the availability is given and bounded at every site, and the problem of every site is a convex program, Kakutani’s Theorem can be invoked, and existence of an equilibrium is easily established. However, uniqueness of equilibrium and the design of a convergent algorithm require further attention. Not surprisingly, the properties of depend on the nature of the distortion.

(1) **Ad valorem tax:** If outflows are subjected at selected sites to a fixed ad valorem tax of arbitrary sign, the equilibrium is unique, and Proposition 1 applies, for payoff functions defined as:

\[
U_s(y_s, d_s) = \max_{0 \leq c_s \leq \bar{c}_s} (c_s) f_s(c_s) + \psi_s^T k_t s(c_s, d_s) - \bar{c}_s^T y_s \\
\text{subject to} \\
\bar{y}_s(c_s, d_s) = y_s.
\]

(6.3)

Here we only allow for imperfections in payments for outflows but fixed taxes or subsidies on internal use can be incorporated in a similar way.

(2) **Prohibitive tax.** If zero returns \( \pi_s = 0 \) make it profitable for the site concerned to stop all outflows, the Nash equilibrium with zero-returns is unique and Proposition 1 applies, for ad valorem taxes set at a prohibitively high level. This is the case independently of whether these distinct sites enclose a territory or not. If this is the case, the territory constitutes a profit-maximizing entity, that takes its inflows as given. However, the model specified in section 4 is not likely to fulfill this requirement, because outflows are a mere byproduct of economic activity, and because there is at any rate an incentive to dispose of the salt.

(3) **Ordered districts.** The simplest case is where the basin can be partitioned into districts that (a) plan consistently by solving an undistorted welfare program, for given inflows, without any payments or fines on flows leaving the district, and (b) can be ordered in sequence, I, II, III, ..., with flows from II not accruing to I, from III not to I or II, etc., that with a delivery matrix \( A \) between territories that can be written in upper triangular form. Then, the Nash equilibrium is unique and can be obtained by applying the co-operative procedure of Proposition 1 to the territories in their order in the sequence, i.e. with zero price for outflows to territories that do not pay. Even though the Jordan basin to a large extent fulfils this condition, the subdivision into national territories at many locations does not follow the relief that governs hydrological flows.

(4) **The general case.**

We now consider the more relevant case with districts of arbitrary shape, possibly even consisting of distinct and non-adjacent territories, which is characterized by zero-payments from specific sites to all its downstream neighbors. We denote the subset of these sites by \( S_o \). Indeed we even abstract from the concept of district altogether and merely distinguish these sites \( S_o \), which may be dispersed and do not necessarily form the boundary of any district. Since the equilibrium conditions are no longer the first-order conditions of an independent welfare program, we formulate the model in decentralized as the maximizing behavior of every site, subject to commodity balances that link the water flows:
where \( \delta_s = 0 \) for \( s \in S^c \) and 1 otherwise. Prices and flows follow from

\[
\hat{p}_{irs} \in \partial d_{irs} V_s, \\
q_{irs} = \sum_r \frac{1}{(1 + \rho_{irs})} y_{irs} \alpha_{rs},
\]

and the equilibrium condition is:

\[
\hat{p} = \hat{p}.
\]

Clearly, the only distinction from the co-operative formulation is the possibility of non-payment \( \delta_s = 0 \). Evaluating this model for given \( \hat{p} \) is not an issue as the standard procedure of top-to-bottom calculations applies. Proving convergence of the iteration process, and uniqueness of equilibrium requires further restrictions, however.

The expression of the sequence of top-to-bottom calculations as the solution of a tax-ridden program can proceed as before, except that every site that does not receive payment for its outflows (which could be negative in view of the penalization for salinity), denoted by the set \( S^c \) is supposed to pay an additional tax \( \pi_{irs}(\hat{p}') \) on outflow, which benefits the downstream sites that receive the flow. Hence, (5.9) is modified to:

\[
U(d^t, \bar{d}^t, y^t, \bar{y}^t, \hat{p}^t) = \max_{q_{irs} \geq 0, d_{irs}, y_{irs}} \sum_S U_s(y_s, d_s) - \sum_{s \in S^c} \sum_t \pi_{irs}(\hat{p}') y_{irs}
\]

subject to

\[
d_{irs} = q_{irs} + b_{irs} \quad (\hat{p}_{irs}) \\
q_{irs} = \sum_r \frac{1}{(1 + \rho_{irs})} y_{irs} \alpha_{rs} \quad (p_{irs}) \\
d^t_{irs} \leq d_{irs} \leq \bar{d}^t_{irs} \quad (\xi^-_{irs}, \xi^+_{irs}) \\
y^t_{irs} \leq y_{irs} \leq \bar{y}^t_{irs} \quad (\zeta^-_{irs}, \zeta^+_{irs})
\]

which relates to the given \( \hat{p}' \) according to:

\[
\hat{p}' = \hat{p}' - (\xi^{+\prime} - \xi^{-\prime}) \quad (6.9)
\]

while

\[
\xi^{+\prime} = \max(\xi^t, 0), \xi^{-\prime} = \max(-\xi^t, 0), \xi^{+\prime} = \max(\xi^t, 0), \xi^{-\prime} = \max(-\xi^t, 0),
\]

\[
\zeta' = \pi(\hat{p}') - \pi(\hat{p}'),
\]

and

\[
d^t_{ks} = d_{ks} \text{ if } \xi^{+\prime}_{ks} > 0 \text{ and } d^t_{ks} = d_{ks} + \kappa \text{ otherwise,} \\
d^t_{ks} = d_{ks} \text{ if } \xi^{-\prime}_{ks} > 0 \text{ and } d^t_{ks} = d_{ks} - \kappa \text{ otherwise,} \\
y^t_{ks} = y_{ks} \text{ if } \xi^{+\prime}_{ks} > 0 \text{ and } y^t_{ks} = y_{ks} + \kappa \text{ otherwise,} \\
y^t_{ks} = y_{ks} \text{ if } \xi^{-\prime}_{ks} > 0 \text{ and } y^t_{ks} = y_{ks} - \kappa \text{ otherwise,} \quad (6.11)
\]
\[ y_{ks}^t = y_{ks} \text{ if } \xi_{ks} > 0 \text{ and } y_{ks}^t = y_{ks} + \kappa \text{ otherwise.} \]

Thus, the price adjustment rule is as in (5.13):

\[ p_{1t} = \rho \left( \frac{1}{\sigma} \right) p_t + \sigma p_t, \quad t = 0, 1, \ldots \] (6.12)

Note that since the tax is only imposed at boundaries of districts, the surface of these districts and the values of the flows through them will become negligible as the grid size is reduced. Indeed, for infinitesimally small cells, the value has measure zero.

**Proposition 4 (Equilibrium with missing markets):** Let the assumptions \( A, WR, WOS, SO, WSC \) and SD hold. Assume, moreover, that at all prices \( p_t \) under consideration the penalization is negligible: \( \sum_{s \in S_0} \sum_{i} \sum_{\tau} \tau_{irs}(p_t) h_{irs} \leq \varepsilon \sum_{s} U_s(y_s, d_s) \), for sufficiently small \( \varepsilon \) modulo \( \sigma \), then for positive \( \sigma \) small enough and less than unity, homotopy process (6.12) converges to the equilibrium of model (6.4)-(6.7).

**Proof.** Part 1. The implementation is not an issue, since in every site-specific program, it is easy to drop the receipts from sales to downstream sites. Part 2. The convergence argument of part 2 of the proof of Proposition 1 holds, since the improvement from bound relaxation in the direction of proportionate reduction of all taxes is achieved for a stepsize \( \sigma \) that is uniformly bounded from below. Hence, for \( \varepsilon \) small enough the possible adverse movement induced by a change in penalty cannot undo the improvement in welfare, and the sequence converges as before.

The above procedure offers a flexible framework for policy simulation with missing markets, provided the grid is sufficiently fine to support the negligibility assumption. It supports the representation of districts of any shape, and hence of any coalition of territories within the basin, who refuse to pay to downstream districts. Furthermore, it permits to represent any geometry of sites (lines, dots, closed shapes) at which the market is missing, i.e. no payments are received, so as to depict imperfections in the monitoring system.

In case the grid is too coarse for this procedure to converge, it still is possible to conduct a sequence of approximations, whereby the penalization in the objective of (6.8) is kept fixed at \( \pi_{irs}(p) \).

### 6.2 Intertemporally efficient path and steady state

The procedure for a single year of simulation is, in principle easily run for several years in sequence, as the future can be considered to lie downstream of the present and the bottom-to-top price calculations to evaluate the selling prices can therefore also be conducted from the last period backward. Hence, the computational procedures sketched so far apply for any simulation over a finite horizon, and the co-operative solution will yield an intertemporally efficient accumulation path.

The steady state can be looked at as an infinite sequence of such calculations, whereby the initial stock coincides with the end stock, and consequently, the initial marginal value of stocks agrees with the end-value. As it clearly is not possible to run infinite sequences, we must compute a steady state in a different way, and since all simulation years are identical, it is natural to address a single-year problem, where we seek to adjust the end prices \( \psi_{1Ts} \), at the end of month
12 of the current year, as well as the initial stocks $k_{i0s}$, the end-stocks of month 12 of the previous year, until in a fixed point is reached at which:

$$k_{i0s} = k_{iTs}, \text{ and } \psi_{i0s} = \mu \psi_{iTs},$$

$$W^*(\psi_T, k_0) = \max_{c_s, d_{iTs}, g_{iTs}, q_{iTs}, y_{iTs}} \sum\left( f_s(c_s) + \mu \sum_i \psi_{iTs} k_{iTs}(c_s, d_s, k_0) \right), \quad (6.13)$$

where $\mu$ is a given discount factor over 12 months, $\psi_{i0s}$ is the current stock price at the beginning of the year, $\psi_{iTs}$ is the current stock price at the beginning of next year, $k_{i0s}$ the stock level at the moment, and $k_{iTs}$ the stock level at the beginning of the next year, i.e. $T$ months later, and we recall from Assumption P that $f_s$ was already discounted to the beginning of the period. Formally, the steady state poses a saddlepoint problem, which can be obtained from the welfare program (5.2) by treating the end-price $\psi_T$ and the beginning stock $k_0$ explicitly as variables:

$$W^*(\psi_T, k_0) = \max_{c_s, d_{iTs}, g_{iTs}, q_{iTs}, y_{iTs}} \sum\left( f_s(c_s) + \mu \sum_i \psi_{iTs} k_{iTs}(c_s, d_s, k_0) \right)$$

subject to

$$c_s \leq \tilde{c}_s(d_s)$$

$$y_{iTs} = y_{iTs}(c_s, d_s, k_0)$$

$$d_{iTs} = d_{iTs} + b_{iTs} \quad (p_{iTs})$$

$$q_{iTs} = \frac{1}{1 + r_{iTr}} y_{iTr} a_{rs},$$

where $W^*$ will now be convex in $\psi_T$ and concave in $k_0$, and the steady state solves:

$$\bar{W} = \min_{\psi \in \Psi} \max_{k_0 \geq 0} \{ W^*(\psi, k_0) - \psi_0 k_0 \} . \quad (6.15)$$

with $\psi_0 \dagger$ denoting the vector-transpose, for given price bounds $\Psi = [\psi, \overline{\psi}]$, where water has a zero lower bound, and a positive (arbitrarily large) upper bound, and salt a large negative lower bound and a zero upper bound. The accepted way to solve such problems is to apply the Arrow-Hurwicz (Arrow et al.,1958) saddlepoint (gradient-) algorithm that adjusts prices and stocks in parallel:

$$\psi_0^{t+1} = \min(\max(\psi_0^t - \overline{\sigma}(k^t_T - k^t_0), \psi_0), \overline{\psi}) \quad (6.16)$$

$$k_0^{t+1} = \max(k_0^t + \overline{\sigma}(\psi_T^t - \psi_0^t), 0), \quad t = 0, 1, \ldots,$$

For positive stepsize $\overline{\sigma}$ chosen small enough, this iteration converges globally to a stationary point. The Envelope Theorem implies that $\frac{\partial W^*(\psi_T, k_0)}{\partial \psi_T} = k_T$ and $\frac{\partial W^*(\psi_T, k_0)}{\partial k_0} = \psi_0$ and hence that the stationary point is a steady state. For positive stepsize $\overline{\sigma}$ chosen sufficiently small, this iteration converges globally. Indeed, with a stepsize sufficiently below the reduction factor
σ of (5.13) or (6.12), the adjustment can be conducted in parallel with the solution of the (distorted) welfare program.

Markov chain

We remark that the fixed shares $\alpha_{rs}$ could be interpreted as the probability of transition of a drop of water from state $r$ to state $s$ along a Markov chain with an annual cycle. Hence, even the deterministic welfare program has a stochastic interpretation. Naturally, further stochastic elements can be incorporated by allowing for a branching of future periods into mutually exclusive classes of states, i.e. for a stationary process by allowing for a third, non-spatial subscript $u$ that defines shares $\alpha_{rsu}$.

Under this interpretation the tax on outflows of Proposition 4 operates as a site-specific (infinitely large) rate of discount, similar to a physical loss that, however, only influences valuation. Consequently, the equilibrium with missing markets can be interpreted as a social plan with a specifically strong discounting profile, and the resulting plan is, because of the convexity of the program, unique and efficient for this given discounting profile.

6.3 Uninhabited sites

In most applications, and the Jordan basin is no exception, the sites will specialize in two groups: markets and links, or populated and deserted sites. Markets are sites with non-negligible production or consumption of the commodities under study. Other sites are links. They have no population, their production function $f^o(c)$ is zero, and they have a single, natural landuse type. Nature reserves under human control are to be treated as inhabited. Though uninhabited, these sites do not actually engage in economic transactions, they transmit the commodities to downstream locations, and price signals to upstream locations.

The distinction is also relevant for numerical reasons, because sites with a low degree of economic activity have little capacity to buffer supply shocks, and may, therefore, cause excessive price fluctuations during the calculations that slow down convergence. Moreover, it seems unnecessary to spend much effort in calibrating supply and demand at these locations, and the same holds for computation time.

For uninhabited sites, the question may be raised whether the convexity requirement on the site-specific model is necessary. The requirements in the proofs of the earlier propositions indicate the following.

(1) Uninhabited sites that receive no inflows from inhabited ones, can adopt any model formulation. This is because these sites act as purely exogenous modules.

(2) At uninhabited sites that receive no inflows from inhabited ones, as human intervention is lacking, it is not meaningful to adjust initial stocks in a steady state calculation.

(3) Uninhabited sites that receive positive inflows should maintain a convex specification with a single landuse type. This is because they actually belong to the welfare program’s technology and cannot be separated as exogenous modules.

Finally, we remark that once uninhabited sites are accommodated for, it becomes natural to operate on a fixed raster of sites. Supposing that the surface of these sites is small enough, we can
also drop the loss factor $\rho_{iTs}$ on flows leaving the site, which amounts to treating all losses as occurring within the site.

**Fluidity**

So far, the specification assumed that all flows of a given month run through the full system within the month under consideration. This was reflected in the month independence of the shares $\alpha_{r}$s. This is highly unrealistic as the time spent in a cell depends on various physical factors but is definitely positive. In other words, the task is to represent the fluidity alluded to in the introduction. The assumption was only maintained for ease of exposition and notation, and the necessary modification is readily accounted for. We define time-dependent shares, $\alpha_{r^{t}t^{t}r}$, with $\alpha_{r^{t}t^{t}r} = 0$ whenever $t > t'$ or $r$ has altitude inferior to $s$, and $\sum_s \sum_{t'} \alpha_{r^{t}t^{t}r} = 1$. The shares define the delays in outflows. The flow equations become for $i = 1,4$:

$$q_{i^t r,t^t} s = \sum_{r'} \sum_{t'} \frac{1}{(1 + \rho_{r^t r^t r})} y_{i^t r^t r^t} \alpha_{r^{t}t^{t}r}$$

and the price equations

$$\pi_{i^t r,t^t} s = \frac{1}{(1 + \rho_{r^t r^t r})} \sum_{r'} \sum_{t'} \alpha_{r^{t}t^{t}r} p_{i^t r^t} r^t.$$  

As a rule, given the small grid size used, inflows will only depend on outflows of the current and previous month. Furthermore, to prevent excessive data storage requirements, the (huge) matrix of outflow coefficients will have to be parametrized in some way, for example, as:

$$\alpha_{r^t r^t r} = \begin{cases} a^0_{r^t r} & \text{if } r \in M^{-}_s, \ t = t' \\ a^t_{r^t r} & \text{if } r \in M^{-}_s, \ t = t' - 1 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, the definition implies that the initial year of simulation should take the inflows that originate from the previous year as given, and similarly for prices in the year $T + 1$.

**6.4 Enhancing the algorithm’s performance: adjustment of reduction factor and imposition of further restrictions**

Within the general setting of processes (5.13), (6.12), we must specify further practical restrictions, so as to obtain an efficient algorithm.

First, it is practical to restrict in (6.9) the search for optimal prices to the compact interval $\Psi = [\psi, \tilde{\psi}]$ of (6.16). This amounts to projecting prices on $\Psi$, via min-max operations as in (6.16), and can be interpreted as allowing for purchases of water and disposal of salt at the bound-prices. Eventually, this price bound should in the optimum be ineffective at all sites, but it is very helpful at the stage of model calibration. The source of anomalies is more easily located, because by keeping the price interval narrow, spatial diffusion of errors is prevented.

Second, we must determine the reduction factor $\sigma$. The simplest approach would be to accept only stepsizes that raise social welfare $U$, i.e. to cut the factor whenever the going value proves unable to reduce this norm. However, often is more practical to work with a norm that allows for improvements in one site to compensate deterioration in another. Therefore, we base
the decision on the $\ell_1$-norm, and only decide to cut after a few unsuccessful iterations, so as to test for iterated contraction.

Finally, we remark that a small parametric variation will, under policy simulation, lead to a small change in endogenous variables. Hence, when initialized at the earlier optimum, the algorithm may need a large number of iterations before the effect of the parametric changes has permeated through the economy, and found expression in a significant aggregate deviations. Consequently, when restarted at an earlier optimum the algorithm might signal convergence and terminate too soon. Perturbing the optimum at the beginning of every scenario simulation can prevent this.

### 6.5 Concluding observations

Is the convexity requirement a serious limitation?

#### Appendix A: Algorithm for optimal portfolio composition

We present two algorithms for solving the optimal portfolio composition problem (5.19).

\[
\Pi^*(\hat{p},d_s) = \max_{w_{sj} \geq 0} \{ \sum_j w_{sj} \Pi_{sj}(\hat{p},\hat{p}_{sj}(\hat{p},d_s)) - C_s(w_{s1},...,w_{sJ}) \mid \sum_j w_{sj} = 1 \}. \tag{A.1}
\]

which we rewrite for short as:

\[
N_s(\Pi_s) = \max_{w_{sj} \geq 0} \sum_j \Pi_{sj} w_{sj} - C_s(w_{s1},...,w_{sJ})
\]

subject to

\[
\sum_j w_{sj} = 1, \quad (\mu_s) \tag{A.2}
\]

with first-order conditions:

\[
\Pi_{sj} - \mu_s \leq \frac{\partial C_s(w_{s1},...,w_{sJ})}{\partial w_{sj}} 
\]

\[
\sum_j w_{sj} = 1. \tag{A.3}
\]

To find these optimal shares $w$, we can use a primal, a dual approach or a mix of both. The dual approach postulates a convex, non-decreasing and differentiable selling-price function $N_s(\Pi_s)$, and retrieves the fractions, via the Envelope theorem, as derivatives. It already follows from the strict quasiconvexity of the cost function that these fractions are unique, and continuous in $s$. However, it is not straightforward to ensure that the derivatives sum to unity. Therefore, we construct the net surplus function through a mixed primal-dual approach, making use of a related program that can be solved in closed form, that is without iteration, and relies on Gorman’s polar form (see e.g. Blackorby et al., 1978). This will enable us to solve (A.2) by iteration over the scalar $\mu_s$, and under further restriction even to solve it in closed form that is without iteration.

We start with the observation that for a given strictly quasiconvex joint output function $C_s$ that is homogeneous of degree one and increasing, we can specify the value function of revenue maximization:
\[ C_s^*(\phi_s) = \max_{v_s \geq 0} \sum_j \phi_{s_j} v_{s_j} \]
subject to
\[ C_s(v_{s1}, \ldots, v_{sJ}) \leq 1 \]  \( (R_s) \)

where in our case \( \phi_{s_j} = \Pi_{s_j} - \mu_s \), and for a given closed form specification of this value function, using the envelope theorem (Varian, 1994) in the standard way, recover the “supply” function:

\[ v_{sr}(\phi_s) = \frac{\partial C_s^*(\phi_s)}{\partial \pi_{sr}} , \]  \( (A.5) \)

in closed form as well. We note that because of homogeneity of degree zero, the relationship \( R_s(\mu_s) = C_s^*(\pi_s(\mu_s)) \) holds. Moreover, it follows that if we can find a value \( \mu_s \) such that \( R_s(\mu_s) = 1 \), supply will be positive for some \( r \) and the shares

\[ w_{sj}(\mu_s) = \frac{v_{sj}(\mu_s)}{\sum_j v_{sj}(\mu_s)} , \]  \( (A.6) \)
are optimal in (A.2), and correspond to the intensities:

\[ v_{sj}(\mu_s) = \frac{w_{sj}(\mu_s)}{C_s(w_{sj}(\mu_s))} . \]  \( (A.7) \)

It also follows that \( w_{sj} = 0 \) is optimal for all \( j \) if \( R_s(0) \leq 1 \). Alternatively, if \( R_s(0) > 1 \), we can raise \( \mu_s \) until we find a value \( \overline{\mu}_s \) such that \( 0 < R_s(\overline{\mu}_s) < 1 \). Generally, \( \overline{\mu}_s = \max_j \Pi_{s_j} - \varepsilon \) will be such a value, for \( \varepsilon \) arbitrarily small. Now we can for any \( \mu_s \) on the interval \( M_s = [0, \overline{\mu}_s] \). The next proposition shows that this can be done by Newton-Raphson iteration.

**Proposition A.1 (portfolio selection: general case):** Consider program (A.2), where the cost function satisfies Assumption PC, and, associated to (A.2), program (A.4). Suppose that positive value can be generated; \( R_s(0) > 1 \); and that there is some given value \( \mu_0 \) for which value is positive but profit negative: \( 0 < R_s(\overline{\mu}) < 1 \). Then, Newton-Raphson iteration with update of bounds \( \underline{\mu}_k, \overline{\mu}_k \) in the process:

\[ \mu^{k+1} = \min(\max((1 - R_s(\mu^k))/\frac{\partial R_s}{\partial \mu} + \mu^k, \underline{\mu}_k), \overline{\mu}_k) , \quad k = 0, 1, \ldots , \]  \( (A.8) \)

converges to the unique fixed point \( \mu^* \in (0, \overline{\mu}) \).

Proof. First, since the cost function is increasing, program (A.4) is bounded and reaches its optimum, which is unique, by strict quasiconvexity. Second, we prove that (A.8) maps into the strict interior of the interval. At \( \mu = 0 \), since by assumption \( R_s(0) > 1 \), it maps strictly into \( M \).
Alternatively, at \( \mu = \overline{\mu} \), we have \( R_s(\overline{\mu}) < 1 \) which also points to the inside. Moreover, by the envelope theorem, \( \frac{\partial R_s}{\partial \mu} = -\sum_j v_{sj} = \frac{1}{C_s(w)} \), which is bounded for \( \alpha \) adding up to unity. Hence, (3.9) converges globally, to a unique fixed point \( \mu^* \), and \( R_s(\mu^*) = 1 \); see also Ortega and Rheinboldt (1970).

We observe that because the function \( R_s(\mu) \) is available in closed form, the fixed-point evaluation proceeds purely by a series of evaluations of this function. Until the fixed point has been found, there is no need to evaluate the supply functions (A.7).

As an illustration, we consider, dropping the subscript \( s \) for notational convenience, the convex function of CES-“plus” type:

\[
C(w_1, \ldots, w_J) = \sum_j \tau_j w_j + G(w_1, \ldots, w_J),
\]

for direct costs \( \tau_r \) and \( G(w_1, \ldots, w_J) = \kappa(\sum_j (w_j)^c)^{\frac{1}{c}} \) with substitution coefficient \( c > 1 \). The associated dual reads:

\[
\begin{align*}
G^*(\phi) &= \max_{\phi \geq 0} \sum_j (\phi - \tau_j) v_j \\
\text{subject to} & \\
\kappa(\sum_j (v_j)^c)^{\frac{1}{c}} &= 1 \quad (R)
\end{align*}
\]

The calculation proceeds as follows. First, we determine the two largest (possibly negative) values of \( (\Pi_j - \tau_j - \kappa) \), say, \( j_1 \) and \( j_2 \). If \( j_2 \), the second largest, is negative, assign \( w_{j_1} = 1 \) and \( 0 \) to the other \( j \)-values, even if \( j_1 \) is negative. If it is positive, use this as lower bound for \( \mu \) in (A.8). We determine ratios, relative to the landuse \( j^* = j_1 \) with maximal revenue:

\[
\tilde{z}_j(\mu) = \frac{v_j}{v_{j^*}} = \left( \frac{\max(\Pi_j - \tau_j - \mu, 0)}{\max(\Pi_{j^*} - \tau_{j^*} - \mu, 0)} \right)^{\frac{1}{c-1}},
\]

which can be simplified to:

\[
\tilde{z}_j(\mu) = g(\mu)z_j(\mu),
\]

for \( z_j(\mu) = \max(\Pi_j - \tau_j - \mu, 0)^{\frac{1}{c-1}} \), and permits to determine the shares as

\[
w_j(\mu) = \frac{z_j(\mu)}{\sum_j z_j(\mu)},
\]

while

\[
R(\mu) = \frac{1}{G(z)} \sum_j (\Pi_j - \tau_j - \mu) z_j(\mu),
\]
\[
\frac{dR}{\mu} = -\frac{\sum_j z_j(\mu)}{G(z)}, \quad (A.15)
\]
and
\[
C(w) = \frac{\sum_j r_j z_j + G(z)}{\sum_j z_j}, \quad (A.16)
\]
since \(\frac{z_r}{G(z)} = \frac{v_r}{G(v)} = \frac{w_r}{G(w)}\) and \(G(v) = 1\). The key property is that \(w(\Pi)\), (where \(\tau\) is a vector with unit elements) is continuous, since \(\max_j(\Pi_j - r_j - \mu)\) is positive, by construction of the upper bound in Proposition A.1.

**Exact algorithm**

We now impose a further restriction on the parameter \(c\) that will enable us to calculate the optimum exactly (up to machine precision), and within a finite number of iterations. Such a property is very useful in our case, because the optimal routing is to be evaluated at every site \(s\), and at every iteration of the price adjustment to be specified below. Finite and exact termination enhances the speed of computations, but, more importantly, it avoids numerical imprecision, which in the ordered chain of calculations over sites, might accumulate and compromise convergence of the overall price adjustment. In case the most rewarding \(j_1\) landuse has negative returns, we assign \(w_{j_1} = 1\) and 0 to the other landuses.

For the alternative case, the finite property is obtained if we choose \(c = 2\), i.e. use the weighted sum of an \(\ell_1\) (absolute value) and an \(\ell_2\) (Euclidean) norm as measure of the transport cost. We now have:

\[
z_j(\mu) = \max(\Pi_j - r_j - \mu, 0) \quad (A.17a)
\]
and
\[
G(z) = \kappa(\sum_{r_j(\Pi_j - r_j - \mu) > 0}(\Pi_j - r_j - \mu)^2)^{\frac{1}{2}}. \quad (A.17b)
\]

Consequently,

\[
R(\mu) = \frac{1}{\kappa} \left(\sum_{j(\Pi_j - r_j - \mu) > 0}(\Pi_j - r_j - \mu)^2\right)^{\frac{1}{2}}, \quad (A.18)
\]
and hence at \(R(\mu) = 1\):

\[
\sum_{j(\Pi_j - r_j - \mu)}(\Pi_j - r_j - \mu)^2 = \kappa^2, \quad (A.19)
\]
which defines a nonlinear equation in \(\mu\).

From (A.17a) follows that all landuses whose returns \(\Pi_j - r_j\) exceed the reference price \(\mu\) will receive a positive allocation. Hence, we can rank the landuses in decreasing order of returns, with \(j_g\) pointing to the landuse of rank \(g\), and \(\bar{h}\) denoting the number of landuses with positive returns. This defines the series of quadratic equations, \(h = 1, \ldots, \bar{h}\), for \(\Pi_{j_g} - r_{j_g} \geq \mu\):
$$\sum_{g=1}^{h}(\Pi_{jg} - \tau_{jg} - \mu)^2 = \kappa^2$$ \hspace{1cm} (A.20)

or

$$h\mu^2 - 2\mu\beta_h + \gamma_h = 0$$ \hspace{1cm} (A.21)

where \( \beta_h = \sum_{g=1}^{h}(\Pi_{jg} - \tau_{jg}) \) and \( \gamma_h = \sum_{g=1}^{h}(\Pi_{jg} - \tau_{jg})^2 - \kappa^2 \), and yields the pair of positive roots \( \mu_h = \frac{\beta_h \pm \sqrt{\beta_h^2 - h\gamma_h}}{h} \), the largest of which is not feasible since it violates the condition \( \mu_h \leq \min_{j} |\Pi_j - \tau_j| - \mu_h > 0(\Pi_j - \tau_j) \). Hence, we only need to consider:

$$\mu_h = \frac{\beta_h - \sqrt{\beta_h^2 - h\gamma_h}}{h}.$$ \hspace{1cm} (A.22)

The algorithm proceeds as follows. It starts with an initial profitability check whether the return on the most rewarding landuse \( (h = 1) \) is not less than \( \kappa \). If this is not the case, all shares are set to zero, as no outflow is profitable. Otherwise, some outflow is possible without losses. To identify the landuses, we test the positiveness of the discriminant \( \beta_h^2 - h\gamma_h \), starting from \( h = \bar{h} \), the full number of landuses with positive returns. If this discriminant is non-negative, evaluate \( \mu_h \) from (A.22) and check whether the inequality \( \Pi_{jh} - \tau_{jh} \geq \mu_h \) holds. Expression (A.22) establishes that \( \mu_h \) will be less than the mean returns \( \beta_h / h \) but for arbitrary \( h \), this value, which is a corrected average, could still exceed the minimal returns of individual landuses, in which case these uses should be cancelled. If both conditions are fulfilled, accept \( \mu = \mu_h \); otherwise update \( h := h - 1 \), which eliminates the least profitable landuse, re-evaluate the discriminant, and test it as well as the inequality for the least profitable landuse.

Convexity and strict quasiconvexity of the cost function ensures that the stationary point obtained in this way is unique and defines the global optimum. We summarize this in the following proposition.

**Proposition A.2 (portfolio selection: \( \ell_1 + \ell_2 \) case):** Consider program (A.4), for \( \phi_{sj}(\mu_s) = \Pi_j - \mu_s \), where the portfolio cost function satisfies (A.9) with \( c = 2 \), i.e. is a weighted sum of an \( \ell_1 \)- and an \( \ell_2 \)-norm. Computational steps (A.17), (A.13), (A.20)-(A.22) evaluate the optimal routing exactly and in at most \( J \) iterations.

Proof. See earlier discussion. ■

**Appendix B: Preparing the data of the watershed**

**Demarcating the watershed**

So far, we have taken the set of sites \( S \) that constitute the Jordan Basin as given. In this Appendix we provide a simple algorithm to construct this set (see also ** for similar algorithms). A watershed is defined as the set of sites that could, if available, deliver water through gravity driven flows, to a given set of client sets or destinations \( \Omega \). The watershed is the catchment area of the client set.
Indeed, a watershed cannot be defined on the basis of an elevation map alone and also requires a given set of destinations, sinks or outlets, that in fact constitutes the origin of the set. From that origin it is possible to identify uniquely the points that could deliver to it. Specifically, suppose that the distribution constants satisfy:

\[
\gamma_{rs} = \kappa \text{ if } a_r \geq (a_s + \varepsilon) \text{ and } s \in N_r^- \text{ and } 0 \text{ otherwise,}
\]

where as before \( a_s \) is the altitude, \( N_r^- \) is the set of downstream neighbors and \( \varepsilon > 0 \) the minimal altitude difference to maintain the flow. We recurrently can determine the watershed set by the procedure:

1. \( m_r = 1 \) for all \( r \in \Omega \);
2. looping over sites \( \ell = 1, \ldots, S \), ranked by increasing altitude \( (a_{r_\ell} \leq a_{r_{\ell+1}}) \):
   
   \[
   m_{r_\ell} = 1 \text{ if for any } s \in N_{r_\ell}^-, \gamma_{r_\ell s} > 0 \text{ and } m_s = 1 .
   \]

The watershed is now defined by

\[
M(\Omega, a, N) = \{ r \in S | m_r(\Omega, a, N) = 1 \} .
\]

Clearly, it is invariant under extension of the client set to members of the watershed, i.e. the set is idempotent:

\[
M(\Omega, a, N) = M(M(\Omega, a, N), a, N) ,
\]

and inclusive:

\[
\Omega' \subseteq \Omega \text{ implies } M(\Omega', a, N) \subseteq M(\Omega, a, N) .
\]

Furthermore, watersheds defined on the basis of different client sets will in general overlap, unless these sets only consist of points on actual rivers. Consequently, since the possible number of client sets is virtually infinite, watersheds are not suitable as a tool to produce a zoning for a surface.

In practical terms, the more accurate the altitude map, the smaller the client set can be kept. When the altitude map is relatively coarse, say, because the grid cells are large, it is advisable to treat all known river flows as elements of the client set. Then, wherever possible, the procedure will direct the (virtual) flows to these rivers and assure connectedness of the watershed.

**Modifying the ranking**

If the altitude map has sufficient resolution, the shares of flows to downstream locations can be determined in accordance with the differences in altitude, while those to upstream locations are:

\[
\alpha_{rs} = 0 \text{ for } s \in N_r^+ .
\]

Very often the resolution will be insufficient, cause altitude to rise along a riverbed, and, consequently, inhibit downstream flows. In this case it would in principle be possible to obtain...
the relevant flow shares $\alpha_{rs}$ by spatial aggregation as well, but this would lead to a share matrix that cannot be written in triangular form, and hence to a system that cannot be solved recursively from top to bottom and requires iteration to meet the physical balances, which would seem less appropriate for the large scale application under consideration. Therefore, the recursivity must be preserved, and since it is not possible to deal with every grid cell separately, through generic rules. The following algorithm performs the task.

(1) Initialize $t = 1$, $\kappa = 0$

(2) Rank sites by decreasing altitude: $r_\ell$, $\ell = 1,\ldots,S$ such that $a^t_{r_\ell} \geq a^t_{r_\ell+1}$, where $a^t_r$ denotes the given altitude at site $r$ at the beginning of iteration $t$ of the procedure.

(3) Run over sites in order $r_\ell$, $\ell = 1,\ldots,S$:

(a) Evaluate the water flows:

$$y^t_r = (1 - \sigma_r)(q^t_r + b_r)$$

where $q^t_r = \sum_s y^t_s \alpha_{sr}(a^t_s)$, and $q^t_r$ denotes the inflow and $y^t_r$ the outflow.

(b) Get steepest descent outflow direction, $s$ from:

$$V_r(a^t) = \min_{\alpha_{rs} \in \{0,1\}} \{ \sum_s (a^t_r - a^t_s)\alpha_{rs} \mid \sum_s \alpha_{rs} = 1 \}.$$ 

(c) if $q^t_s > q$ and $V_r(a^t) < 0$ then

if $\left( a^t_s = a^t_x \right)$ then

$$a^{t+1}_s = a^t_r - \epsilon$$

leveling

else

search neighborhood of $r$: $R^t_r = \{ s \in N_r \mid s \neq x, a^t_s \leq a^t_x, a^t_r - a^t_s - \epsilon \geq 0 \}$

if $\left( R^t_r = \emptyset \right)$ then

if $a^t_r = a^t_x$ then

$$a^{t+1}_r = a^t_r + \epsilon$$

filling

else

$$\kappa := \kappa + 1$$

number of trouble spots

else

for $s \in R^t_r$, $a^{t+1}_s = a^t_r - \epsilon$

leveling

endif

endif

if $\kappa \leq \kappa$ go to (2)

endif

go to (4)

(4) End
The procedure can terminate in two ways. Either all trouble spots have been eliminated, in which case condition (c) is never met, or the maximum $\kappa$ on the number of trouble spots which were already filled up has been reached. Hence, the process terminates in a final number of iterations $t$. In the application for the data of the Jordan basin, only 1 trouble spot eventually remained.

**Mini-watersheds.** The demarcation of watersheds may exclude territories surrounded by the watershed. The boundaries of such areas lie below the watershed, and are excluded because they do not contain any client point, nor are they surrounded by it. In fact, these are mini-watersheds, that can be represented within the main one, by defining additional client points. However, it is important to ensure that the

**Canal irrigation.** Canal irrigation represented is through a separate map of downward flows that drain into the land, starting from the tapping point on the river. With a slight modification the leveling procedure can be used to ensure that all irrigated land can be reached, including the return flows to the river. The difference is that in step (c)), leveling and filling is not activated for sites at the fringes of the irrigated zone.

**Lake Tiberias and Transnational Carrier.** As the aim of the modeling exercise is not to study the aquatic resources and flows in the lake, we represent it by a hollow land surface with a single outlet point. At this point the whole water volume of the lake is stored, a fraction of which is extracted for the Transnational Carrier to the West, thereby leaving the watershed.

**Extraction from the Yarmouck river by Jordan and Syria.** The Yarmouck separates Jordan and Syria in the Western part of its bedding. At the Adashiya Dam, water enters into the ECGM canal, that runs through Jordan and is the source of water for the (pump-based) conveyor to Amman, where it is discharged on a river within the Jordan basin. In addition, a Northern Conveyer has been planned that pumps from the Yarmouck at an upstream location, and also eventually discharges in the basin. Similar conveyors are already operational in Syria. For our modeling, the difficulty is that the pumped water moves against the gravity order while the non-evaporated fraction remains within the watershed. Our approach can only represent the gravity driven flows in a spatially explicit manner, since upstream flows go against the order of calculation, essentially causing simultaneity in the flow structure. While it is possible to allow for this between annual cycles, and through the introduction of an additional level of iteration, it would be possible to obtain convergence of the simultaneous system as well (because the share matrix is non-expansive), we choose to avoid this because model calibration becomes significantly more difficult. Hence, the net disappearance of water in the conveyors is treated as a tapping that leaves the watershed and, to deal with the return flows, we draw virtual underground canalization that resurfaces at points below the source of the conveyor, where the water re-enters the basin.
References


The Centre for World Food Studies (Dutch acronym SOW-VU) is a research institute related to the Department of Economics and Econometrics of the Vrije Universiteit Amsterdam. It was established in 1977 and engages in quantitative analyses to support national and international policy formulation in the areas of food, agriculture and development cooperation.

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