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Pricing a raindrop in a process-based model: Theory and application for the Upper-Zambezi

by

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Abstract

A general approach is presented to value the stocks and flows of water as well as the physical structure of the basin on the basis of an arbitrary process-based hydrological model. This approach adapts concepts from the economic theory of capital accumulation, which are based on Lagrange multipliers or shadow prices that reflect market prices in the absence of markets. This permits us to derive a financial account complementing the water balance in which the value of deliveries by the hydrological system fully balances with the value of resources, including physical characteristics reflected in the shape of the functions in the model. The approach naturally suggests the use of numerical optimization software to compute the multipliers, without the need to impose an immensely large number of small perturbations on the simulation model, or to calculate all derivatives analytically. A novel procedure is proposed to circumvent numerical problems in computation and it is implemented in a numerical application using AQUA, an existing model of the Upper-Zambezi River. It appears, not unexpectedly, that most end value accrues to agriculture. Irrigated agriculture receives a remarkably large share, and is by far the most rewarding activity. Furthermore, according to the model, the economic value would be higher if temperature was lower, pointing to the detrimental effect of climate change. We also find that a significant economic value is stored in the groundwater stock because of its critical role in the dry season. As groundwater comes out as the main capital of the basin, its mining could be harmful.

Keywords: Process-based models, river-basin management, capital theory, valuation, simulation, optimal control, decision support system.

Section 1
Introduction

The international community has come to recognize that fresh water is scarce, witness the declarations at the Dublin Conference in 1992, the World Water Forum in 2000 and the UN conference Johannesburg 2002. It is generally felt that the problem is not so much physical scarcity, as inefficient resource management and use, hence the need to consider water as an economic good, specifically as a renewable resource.

In response to these perceptions, hydrologists have shifted their traditional focus on physical engineering aspects, to include socio-economic as well as environmental impacts in their dynamic, process-based simulation models. There is even a class of models, known as Integrated Assessment Models (IAM), that link these domains. However, the integration is weak. Whereas the IAMs built by economists are not process based and, therefore, only provide a rudimentary representation of hydrological cycles (Naveh, 2000, Krautkraemer and Batina, 1999), the process-based models built by hydrologists have relegated the socio-economic aspects to separate compartments within the system. Consequently, the process-based models cannot be used to answer normative questions, say, where to locate zones with highest priority for intervention or how to measure resource scarcity, since there is neither a central authority in the model overlooking and weighting all dynamic aspects nor a competitive market generating appropriate price signals.

The present paper applies the valuation methodology proposed in Keyzer (2000) to the Upper Zambezi basin. The method adapts classical cost based pricing techniques from the theory of reproduction (e.g. Morishima, 1973) to process-based models, taking as given the process-based model developed by the hydrological expert. It treats the hydrological process as a large-scale production process at the level of the river basin that produces water at different locations and points in time. To substitute for the absence of any market, an imaginary river-basin authority is introduced, that overlooks the entire situation in the basin and maximizes a given welfare objective which measures the overall benefits generated in the basin. However, this authority takes the entire process including human activity as given and merely evaluates the benefits from marginal changes or perturbations in process parameters. Comprehensive insight into these marginal effects is critical for any understanding of the contribution of various parts of a process to its overall functioning, albeit that by construction a marginal approach cannot account for the consequences of larger changes. Yet, such information is essential for management decisions. Indeed, within commercial firms this is the very *raison-d'être* of internal accounting departments. The internal accountant or the manager needs to determine how far to zoom into the organization or processes and not all processes require the same detailed level of information. So, the accounting methodology should allow for flexibility and the proposed method offers this flexibility with respect to location and time.

These dynamic aspects are of special importance in hydrology, because the consequences of even a minor perturbation may extend over a prolonged time period, for example through their impact on groundwater tables. Capital theory offers the natural vehicle to trace these dynamic implications back to the original event.[1] The impact per unit of perturbation can be calculated as a Lagrange multiplier, or shadow price, of a dynamic optimization problem. These multipliers are referred to as prices, because they would coincide with actual market prices if the underlying resource could be traded on a competitive market. The notable feature is that we implement this relatively standard optimization technique outside its common realm of application, which might seem to disqualify it as a numerical technique for this case.[2] However, process-based model possess the distinctive simplifying feature that they consist of a system of (usually) deterministic, nonlinear difference equations. Consequently, they can be solved through straight simulation without requiring any iterative adjustment. In fact, in process-based models the optimization is only needed to calculate the shadow prices, which is still a nontrivial problem because of the nonlinearity and size of the model.

Specifically, the following problems can be tackled. First, the process based models represent various natural processes such as precipitation and solar radiation, whose valuation is not trivial. By attributing value to every inflow, the approach can price natural endowments such as raindrops or daily temperatures. Second, hydrological models distinguish between stocks and flows. The flows are deliveries to other parts of the system, but stocks transfer value over time and should be priced as well. As they belong to the physical capital of the system, capital theory offers the natural vehicle to price them. Finally, the hydrological model also reflects the physical structure of the basin. Physical characteristics of processes, like slopes and vegetation, contribute in a positive or negative manner to the system's overall value and can in their totality also be looked at as capital. However, in the model they are represented by functional forms rather than by stock levels. Nonetheless, a homogeneity transformation makes it possible to express these characteristics as resource stocks and to value them accordingly.

Besides this transformation that us to attribute value to structural characteristics, another numerical "trick" relates to the valuation of stocks in the steady state. Although capital theory allows us to decompose the optimization problem into a sequence of annual problems, the key question in numerical implementations is how to value the remaining stocks at the final date. In economic modeling a common assumption is to use the steady state value for this, with all exogenous parameters are kept constant at the levels reached at the model's final date. In a

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[1] Capital theory makes extensive use of Bellman's principle and derives the value of the stock from the discounted cumulative value of its optimal use, see e.g., Stokey and Lucas (1989) and chapter 8 in Ginsburgh and Keyzer (2002)

[2] Usually these techniques are used to solve so-called convex optimization problems, and, in rare cases, as parts of algorithms to solve specially structured non-convex problems. By contrast, the process-based models in hydrology are highly non-convex.

hydrological system the steady state is of special relevance because it defines the conditions under which the system is able to reproduce itself forever while preserving all its value generating potentials. This corresponds to the notion of sustainability. In fact, it is this steady state value we shall report on. However, the evaluation of this steady state value of stocks is difficult in this large nonlinear model because the end-price appears as an unknown coefficient in the objective of the optimization program and affects all elements of the valuation. One might expect that it has to be obtained by iterating over different values until equilibrium is found. This would be very demanding. We show that the special structure of the optimization problem permits us to evaluate the objective coefficient directly as a multiplier of a related program without unknown coefficients. The approach is implemented in the AQUA model, an existing process-based hydrological model of the Upper-Zambezi river basin developed by Hoekstra (1998). To concentrate on the hydrological process, we drop the socio-economic relationships of this model, replacing these by a list of end uses, such as monthly deliveries for irrigation, at exogenously given prices.

To sum up, the evaluation of shadow prices can be used to track value flows through the hydrological process. We distinguish four categories of resources that generate value: endowments, stocks, the hydrological structure and water-related economic activity. The paper proceeds as follows. Section 2 introduces the basic microeconomic principles of shadow price based accounting. Section 3 applies these principles in a hydrological context. In Section 4, the steady state version is presented and the numerical procedure described, that is in Section 5 applied to the model of the Upper-Zambezi basin. In Section 6 some concluding remarks are made. This paper also contains several appendices. Appendix A illustrates some of the abstract notions introduced in Section 2. The AQUA model is introduced in Appendix B and three subsequent appendices discuss step-by-step several of the modifications introduced to this mode for performing the valuation method. The last two appendices, Appendix F and G, focus on some general problems that are of relevance for future applications.

Section 2
Accounting, shadow pricing and Euler's rule

To introduce the concepts of accounting and shadow pricing, we review some basic notions from microeconomic theory, e.g. Kreps (1990) and Varian (1996). Appendix A contains numerical examples to illustrate these concepts. Our starting point is a small firm that produces a single output in the amount of d from a net input x, representing net inflows such as labor, energy, water or raw materials. This firm is assumed to maximize its profits while taking as given the market prices p and w of its output and input, respectively. The firm's production technology is described by its production function $d = f(x)$, whose (vector of partial) derivative(s) is denoted by $f_x'(x)$. Supposing that f is concave, the firm's profit maximization problem can be written

$$max_x \; pf(x) - w'x, \tag{2.1}$$

where $w'x$ denotes the inner product of x and w, where w' denotes the transposed vector w. The associated first-order conditions read

$$pf_x'(x) = w, \tag{2.2}$$

i.e., the marginal contribution or product of each input is equal to its marginal cost. The intuition is obvious. Suppose (2.2) holds with a greater than sign for one of the inputs. Then, the firm can increase its profit by slightly expanding this particular input, because the increase in revenue exceeds the additional cost. A profit-maximizing firm will adjust its input until equality holds. This also illustrates that marginal contributions and marginal costs are the drivers of economic behavior, as opposed average productivity or costs. Within hydrology, the use of average costs or average productivity is widespread; see e.g., Droogers and Kite (2001) for an arbitrary but recent example.

The firm's financial *account* or *value balance* has its revenue $pf(x)$ on the left hand side and the purchases of inputs $w'x$ plus net profit on the right. Net profit is the residual $pf(x) - w'x$ that balances both sides and arises from the given structure of the firm's technology and, hence, from the shape of f. The value balance relates the revenue of deliveries to customers (destination) to the sum of input value and net profit (source).

This simple profit maximization problem is easy from a computational point of view, because it corresponds to a convex optimization problem. Such problems generally consist of several in and outputs, a concave objective function and variables x that are restricted to some convex set.[3] However, natural processes such as hydrological processes often impose restrictions upon the variables x that make the set of admissible candidate solutions non-convex. The latter

[3] A convex set is defined by the property that any pair of points in this set can be connected by a straight line segment that also belongs to the set, for example the area inside an ellipse is a convex set.

optimization problems are difficult to solve computationally and this calls for a way to circumvent this problem, which is postponed to section 5.

Besides the occurrence of non-convexities, the hydrological process is also characterized by the fact that it is not possible to adjust all quantities at will. Floods and droughts have to be taken as given. The implication is as follows. Suppose that, in our previous example, all inputs are fixed at $x = \overline{x}$, then the firm's problem becomes

$$\begin{aligned} & max_x \; pf(x) - w'x \\ & subject\ to \\ & x = \overline{x}, \qquad\qquad (l) \end{aligned} \tag{2.3}$$

where l denotes the vector of Lagrange multipliers or shadow prices, which is not necessarily positive. These shadow prices measure the sensitivity in the firm's profit, should the (fixed) amount $\overline{x}$ increase slightly. To see this, rewriting the first-order conditions of (2.3) yields

$$l = pf_x'(\overline{x}) - w. \tag{2.4}$$

The interpretation is immediate. Higher (absolute) values indicate a higher sensitivity to changes in the variables and, if action can be undertaken, it indicates where policies affect the outcome most, illustrating the relevance of this information for policy decisions. Note that the multipliers can become negative, in which case a reduction of the input would increase profits, which corresponds to the idea of a damaging flood or excess precipitation causing pests in agriculture production. Also note that the shadow prices measure the amount the firm would be willing to pay on top of the market prices w for an incremental increase in its input, or, if negative, the subsidy it would require to process this input quantity.

One remark is in order that turns out to be crucial in the large-scale process-based model under consideration. It is easy to see that simple substitution of x by $\overline{x}$ in the objective yields $pf(\overline{x}) - w'\overline{x}$. Thus, the firm's profit can be directly calculated without performing the optimization and, given the constraint, $x = \overline{x}$ is the optimal level for x. However, the substitution does not yield any information about the shadow prices and these can be obtained in three different ways. The most obvious one is to repeat the same substitution in (2.4), which has the disadvantage that it requires analytical expressions for the partial derivatives. The latter can be overcome by performing the optimization in numerical optimization software, at the costs of some programming. The third option, which is also numerical, is to calculate the sensitivity of the profit function to small perturbations in $\overline{x}$, which can be done in any programming language. However, as discussed later on in section 4, this method is sensitive to the size of the imposed perturbations and requires a sequence of perturbations, because each individual input has to be perturbed in isolation. Commercial optimization packages are designed to deal with these problems and are therefore preferable.

Accounting and Euler's rule

Next, suppose that there are several inputs, and that we wish to develop an account that values the marginal contribution of each of these, while positing value exhaustion, i.e. all value should be attributable to inputs and technology. This is not possible if profit is obtained as a residual. First, note that in problem (2.3) the marginal value produced by x equals $w'x + l'x$, while only $w'x$ is paid for. Second, profit also results from the shape of the function *f*, and hence, from the given structure of the technology. The question is how to assign value to this shape. Here we introduce an additional endowment z, a (scalar) variable to be referred to as a *structural factor*, whose shadow price effectuates value exhaustion. The idea is to redefine the function to include z that has no physical effect, and to insert simultaneously an additional constraint within problem (2.1). The redefined production function and the additional constraint become:

$$\tilde{f}(x,z) = zf(x_1/z,\ldots,x_n/z) \text{ and } z = 1. \tag{2.5}$$

The firm's maximization problem becomes

$$\begin{aligned} & max_{x,z}\ p\tilde{f}(x,z) - w'x \\ & \textit{subject to} \\ & \quad x = \bar{x}, \qquad\qquad (l) \\ & \quad z = 1, \qquad\qquad (d) \end{aligned} \tag{2.6}$$

where d denotes the shadow price associated with the last constraint. The value exhaustion follows from rewriting the first-order conditions, which yields:

$$p\tilde{f}_x'(x,1) = w + l \text{ and } p\tilde{f}_z'(x,1) = d\ , \tag{2.7}$$

where $\tilde{f}_x'$ and $\tilde{f}_z'$ denote partial derivatives of $\tilde{f}$ with respect to x and z, respectively. The production function is now homogeneous of degree one.[4] By Euler's rule we then have:

$$\tilde{f}(x,z) = f_x'(x)x + \tilde{f}_z'(x,z)\, z. \tag{2.8}$$

Then, application of Euler's rule in the second equality of (2.7) implies that

$$d = p\tilde{f}_z'(x,1) = p[\tilde{f}(x,1) - f_x'(x)\, x] = pf(x) - w'x - l'x. \tag{2.9}$$

[4] Formally, $\tilde{f}(x,z)$ is homogeneous of degree one if $\tilde{f}(lx,lz) = l\tilde{f}(x,z)$ for all $l \geq 0$. Euler's rule states that any differentiable function $\tilde{f}(x,z)$ that is homogeneous of degree one satisfies $\tilde{f}(x,z) = \tilde{f}_x'(x,z)x + \tilde{f}_z'(x,z)z$.

Thus, the Lagrange multiplier d is equal to the firm's profit as asserted. With respect to the firm's account, by explicitly including d we obtain that revenue $pf(x)$ automatically balances against $w'x$, $l'x$ and d. Net profit equals $l'x+d$ and value exhaustion is automatically effectuated

Accounting subsequent processes

Hydrological systems typically consist of several subsequent processes or functions grouped in compartments and we sketch how the optimization can be used to produce separate accounts for each compartment as well as the aggregated account. For this, suppose that the 'upstream' process described by the function h uses inputs x to produce the vector of inputs y for the 'downstream' process represented by the function g such that

$$f(x) = g(h(x)). \tag{2.10}$$

There are two important issues. First, how to measure the contribution of the processes g and h separately, which boils down to the redefined functions

$$\tilde{g}(h,z_g) = z_g g(h / z_g) \text{ and } \tilde{h}(x,z_h) = z_h h(x / z_h) \tag{2.11}$$

for the structural factors z_g and z_h. The additional constraints are $z_g = 1$ and $z_h = 1$, with multipliers d_g and d_h, respectively. Second, how to price the internal delivery y. This price, denoted as f, is important, because process h delivers y to process g and, therefore, it enters both disaggregated accounts per process. The idea is to introduce the *accounting row* $q = y$ and write $g(q)$ instead of $g(y)$, which also causes no physical effects to the hydrological model. The internal price for y corresponds to the multiplier of the accounting row, because it measures the marginal impact of an incremental change in y on profits. Combining all this means that (2.3) should be written as

$$\begin{aligned}
&max_{x,y,q,z_g,z_h} \; pd - w'x \\
&subject\ to \\
&d = \tilde{g}(q,z_g), \quad z_g = 1, \quad (d_g) \quad q = y, \quad (f) \quad Downstream \\
&y = \tilde{h}(x,z_h), \quad z_h = 1, \quad (d_h) \quad x = \overline{x}, \quad (l) \quad Upstream
\end{aligned} \tag{2.12}$$

The account for downstream process g has revenue pd balancing against virtual spending $f'y$ for its input and net profit d_g generated by this process. Similarly, the account for upstream process h balances revenue $f'y$ for internal deliveries to process g with expenditures on inputs $w'x$ and profit $l'x+d_h$. Finally, the aggregated account of (2.12) adds up these two accounts and yields the same account as for (2.6) if internal deliveries are canceled. The above illustrates how the proposed valuation method makes it possible to zoom in on the specifics of the technological process.

The above illustrates how the proposed valuation method makes it possible to zoom in on the specifics of the technological process. Conversely, it would be possible to drop the distinction between both structural factors and only distinguish a single z, but in this case presentation of separate accounts would be impossible. Hence, the main point to note is that the separation of accounts requires on the one hand that all structural factors be kept distinct, and on the other hand that the processes be linked by accounting rows like $q = y$ that equate the outputs of upstream processes with downstream inputs. In short, the conversion of the process-based model to the form that can generate accounts at the level desired by the modeler or policy maker essentially consists of introducing structural factors and their constraints, as well as accounting rows that *group* or bracket the equations into separate compartments.

A word of caution may be warranted with respect to the incorporation of accounting rows. Suppose that we had written $q = \tilde{h}(x, z_h)$, instead of $y = \tilde{h}(x, z_h)$. Then, the price f would have been zero, despite the fact that all quantities would remain unchanged. The reason is that in this case $q = y$ can only measure the flow q but does not play any role in the transfer of quantities to the subsequent compartment.

The discussion of where to include accounting rows and its correct implementation indicates that the design of a transparent flowchart of the physical flows in the process-based model is an essential step, since it defines the grouping of processes into separate compartments and, hence, it visualizes the places where accounting rows need to be inserted. The latter essentially amounts to the construction of a system of accounts as part of a management information system. It is the most delicate task of the accounting exercise, that requires a comprehensive insight into the model and the underlying real world processes and should, therefore, be based on a dialogue between the process engineers and the ‘internal accountant’. Figure 1 in section 5 below gives a stylized flowchart for the AQUA model.

Section 3
Capital valuation in a process-based model[5]

Capital theory employs the accounting principles introduced in the previous section and was designed to value stocks, flows and the technological structure. In this section capital theory is integrated to process-based hydrological models, but first the mathematical framework of large-scale process-based models needs to be introduced for a better understanding.

Process-based hydrological models used in policy applications generally take the form of a system of deterministic, nonlinear difference equations that are simulated over a finite horizon. Some applications in hydrology postulate differential equations, but their numerical implementation requires a discretization of time, and effectively amounts to a transition to a system of difference equations. In case the model is numerically implemented in simulation software, each statement in the computer code represents one of the model equations.

To represent full cycles – or periods – in process-based models we introduce the time dimension t, $t = 0,1,\ldots$, and define the duration of the cycle as the time necessary for one repetition of the given hydrological pattern of inflows. For most applications, seasonality suggests an annual cycle. Within each cycle, we distinguish stocks and flows by location within the river basin and points in time, e.g., months. The location-date combinations within each cycle are denoted by the index $j = 1,\ldots,m$. For example, in the application of section 5, index j represents five locations on a monthly basis (the topsoil associated with three land cover types, surface water reservoir and one aquifer), and hence comprises sixty elements. Each j could be imagined as representing a 'firm' that passes through input of water from other locations and dates into output of water with the characteristics of location-date j, for example wet season's precipitation differs from dry season's precipitation. Next, the stock of water contained at 'location-date' j within cycle t is denoted as $k_{j,t}$ and, stacked as a m-dimensional vector, $k_t = (k_{1,t},\ldots,k_{m,t})$. Natural endowments or inflows per location j, such as precipitation, temperature and area per land cover type in section 5 are flow variables and denoted by the nonnegative vector g_t (of appropriate length). The inflows within each cycle are kept constant across cycles, i.e., $g_t = w$ for all t and for some vector denoted by w. The n-dimensional vector $d_t = (d_{1,t},\ldots,d_{n,t})$ represents flows induced by human activity in cycle t and the elements of d_t are indexed by $h = 1,\ldots,n$. Human activities include monthly water extraction distinguished by sector and source, as well as return flows from restitution to the hydrological system after use.

More generally, any hydrological process describes how the location-date specific stocks evolve from one cycle to the other due to *(i)* inflow from precipitation and heat, *(ii)* the physical laws of nature such as evapotranspiration and *(iii)* anthropogenic effects, including human extraction and return flows. It can be captured by a vector function H that describes the

[5] This and the following section build on Keyzer (2000).

hydrological process as a system of autonomous difference equations, i.e., H is independent of time, as follows

$$k_{t+1} = H(k_t, d_t, g_t), \qquad t = 0,1,\ldots, \tag{3.1}$$

for given initial stock k_0. Note that in applied studies the model is usually programmed in a software language and program statements can be regarded as the equations of the function H.

Human activities are taken to obey some behavioral rule that depends upon the inflows of resources and the stocks of water available. This means that human activity reacts to the evolution of the hydrological system in a specified manner through functional forms. The simplification is, obviously that it is not supposed to react to endogenous prices. For instance, the scope for irrigation depends on the amount of precipitation, temperatures and the availability of water for this activity and, based upon their experience, farmers will follow some 'rule of thumb' to cope with different situations, which is the functional form in the model that relates irrigation applied to inflows and stocks. The formulation could also allow for a reaction to exogenous policy instruments such as water prices but not to the prices calculated in the valuation. The behavioral rule is described by the function F, which is also taken to be independent of time for convenience, and it specifies

$$d_t = F(k_t, g_t), \qquad t = 0,1,\ldots\,. \tag{3.2}$$

The vector d_t can also be thought of as representing deliveries from the hydrological process to the economy (extraction) for positive values and vice versa (return flows) for negative values.

Simulation of the process-based model means that the variables are evaluated recursively over time. The key property is that this uniquely determines the time path of all variables. To see this, note that given the stocks k_t at the beginning of cycle t, the simulation starts by first solving $g_t = w$, then (3.2) is solved to obtain d_t and, finally, (3.1) yields k_{t+1}. This insight is important, because below $g_t = w$, (3.1) and (3.2) represent the constraints in a dynamic optimization problem and, similar as discussed in section 2, the optimal 'x' variables can be obtained through recursive substitution without performing the optimization, but doing so does not yield shadow prices. Hence, the need to embed the process-based model in an optimization problem.

The optimization problem

When determining the economic value generated in the river basin, we consider the geographical unit of the river basin as one large-scale production technology consisting of many subsequent processes that handle site and date specific amounts of water. In order to evaluate the performance of this large-scale technology an objective function is needed, referred to as the *welfare* function, that measures the total benefits in the river basin. The marginal change in the value of this welfare to perturbations in the model parameters, determines the Lagrange multipliers, or shadow prices. As discussed in section 2, we can conceive of the optimization problem that evaluates these multipliers as describing an imaginary river basin authority that monitors the entire situation from the perspective of a social planner but is perfectly powerless because it cannot change anything in the physical sphere. This planner finds itself in the same situation as the firm in (2.3), whose technology and input level are given. Clearly, the next step would be to let the authority actually implement the perturbations it envisages. We return to this aspect in the last section.

We assume for convenience that as welfare objective the authority uses discounted (net) revenue per cycle from selling the resource to end users, i.e., d_t , over an infinite time horizon at the nonnegative price-vector p across cycles, although any nonlinear, homogeneous function for welfare will do. The price vector is constant across cycles and of the same dimension as d. The value for one cycle (or year) can now be expressed as the inner product $p'd_t$ and the authority maximizes the overall discounted sum $\sum_{t=0}^{\infty}(\boldsymbol{r})^t p'd_t$, where $r < 1$ is a fixed, positive discount factor.

As explained in the previous section, we introduce annual structural factors to redefine the functions H and F and to group processes for accounting purposes. These factors are stacked into the vector z_t and the constraint $z_t = 1$ is added to the model. For notational simplicity, we specify one structural factor per equation in the remainder of this section, i.e., the most disaggregate level possible. Formally, define the $(n + m)$ - dimensional vector of additional structural factors in year t as z_t, with the understanding that the first m variables are associated with the function F and the last n variables with H. Then, the redefined functions for F and H are:

$$\tilde{F}_h(k_t, g_t, z_t) = z_{h,t} F_h(k_t / z_{h,t}, g_t / z_{h,t}), \qquad h = 1, \ldots, n,$$

and (3.3)

$$\tilde{H}_j(k_t, d_t, g_t, z_t) = z_{n+j} H_j(k_t / z_{n+j,t}, d_t / z_{n+j,t}, g_t / z_{n+j,t}). \qquad j = 1, \ldots, m.$$

These functions are now homogeneous of degree one by construction and Euler's rule applies to each line. The shadow prices associated with the constraint $z_t = 1$ have a clear interpretation since these are equal to the net proceeds from the system's operation at its most disaggregated level. If each part of the system could be privatized like an ordinary company, then at the 'stock exchange' the 'firm' at location j would have a value equal to the discounted value of the infinite stream of associated shadow prices. In the absence of markets, it is the authority that imposes prices through shadow pricing. Furthermore, in attributing the eventual value, it will be practical to distinguish between sources and sinks, with positive and negative multipliers, respectively.

We are now almost ready to state the maximization problem faced by the imaginary authority. Yet, as we may also wish to isolate the marginal contribution of the stocks in each year, we treat the stocks present at the beginning of each year as additional variables $\tilde{k}_t$ and impose the accounting row $\tilde{k}_t = k_t$ for all t to determine their shadow prices. Similar, the accounting row

$g_t = w$ is introduced to price natural endowments or inflows such as precipitation. So, for given initial stocks k_0, the mathematical framework for the authority's problem is:

$$V(k_0) = max_{d_t \geq 0, g_t \geq 0, k_{t+1} \geq 0, \tilde{k}_t \geq 0; t=0,1,\ldots} \sum_{t=0}^{\infty} (r)^t p' d_t$$

subject to (3.4)

$d_t = \tilde{F}(\tilde{k}_t, g_t, z_t)$,	(q_t)	*Human behavior*
$k_{t+1} = \tilde{H}(\tilde{k}_t, d_t, g_t, z_t)$,	(m_t)	*Hydrological processes*
$g_t = w$,	(l_t)	*Endowments*
$\tilde{k}_t = k_t$,	(y_t)	*Stocks*
$z_t = 1$,	(d_t)	*Structural factors*

where $V(k_0)$ denotes the value function representing the maximum value attainable within the river basin, and q_t, m_t, l_t, y_t and d_t denote the vectors of relevant Lagrange multipliers. Note also that extracting the information on the multipliers of resources, stocks and the value generating capacity of the hydrological structure comes at the expense of introducing the additional variables $g_t, \tilde{k}_t, z_t$ and the associated equations.

As already mentioned, simulation of the process-based model means that all the variables are solved and uniquely determined. So, the optimal solution for the physical variables in (3.4) can be obtained without performing the optimization. However, retrieving the shadow prices does require optimization.

The multiplier q_t is of special interest since it provides a measure of the divergence between the behavioral rule followed by the end users in the river basin and overall optimal water use from the perspective of the authority. If it is equal to zero, then the behavioral rule is also a local optimum in the extended model where the authority is not powerless.[6] Thus, the Lagrange multiplier q_t quantifies the importance of the welfare distortion caused by the behavioral rule. Since we do not impose a behavioral rule in our numerical application we do not refer to it in the sequel.

Bellman's principle of backward induction

Optimization problem (3.4) is not tractable in any numerical application, because of its infinite horizon that implies an infinite number of variables and an infinite number of constraints. Even a reasonable finite horizon approximation has too many unknowns to offer a tractable framework for computing the shadow prices. However, application of Bellman's backward induction within this setting of capital theory makes it possible to decompose the (in)finite horizon problem into a

[6] Because of the non-convexities in the hydrological processes further investigation has to reveal whether or not the local optimum is also the global optimum.

sequence of smaller single period problems with a finite number of variables and equations. This decomposition yields the same optimal time path as the original (in)finite horizon problem including the shadow prices along this time path. Note that in (3.4) the value function $V(k_t)$ could also be evaluated for subsequent periods, and denotes the maximum attainable welfare attainable in the system from period t onward, given a stock of k_t at the beginning of period t. Then, according to Bellman's principle the function $V(k_t)$ is equal to the maximum of $p_t' d_t + rV(k_{t+1})$ over all the variables in cycle t subject to the constraints in cycle t in (3.4). The difficulty is, obviously, that this function is not known. However, to determine this optimal value, we only need to maximize $p_t' d_t + r y_{t+1}' k_{t+1}$, because for given stock level k_{t+1}, the shadow price y_{t+1} for end-of-cycle stocks measures the contribution of k_{t+1} to future discounted welfare and it therefore equals $V'(k_{t+1})$ in case V is differentiable.

Summarizing the discussion thus far we obtain a sequence of single period decisions, whereby the authority maximizes the current period's return plus the present (or discounted) value of the end-of-period stock. We can now calculate an optimal level of activities by solving for a single cycle t, t = 0,1,…, the optimization problem given by

$$max_{d_t \geq 0, g_t \geq 0, k_{t+1} \geq 0, \tilde{k}_t \geq 0} \; p_t' d_t + r y_{t+1}' k_{t+1} \qquad (3.5)$$

subject to the same set of constraints in (3.4) for cycle t and for given k_t. To solve this problem, the stock k_t should be known, which is the case, since straight simulation yields the time path for all variables. In addition, end values y_{t+1} should also be known. For a finite horizon problem, with known stock value in the last period, these can be obtained by exploiting Bellman's principle of backward induction, whereby for given quantities, the stock values y_t depend on y_{t+1} and k_t. Since Bellman's principle is also valid for the non-autonomous case this makes it possible to conduct valuation also for non-autonomous processes where endowments w and the functions H and F vary over time. We leave the infinite horizon case for the next section and return to the issue of accounting.

From shadow prices to accounts

We are now ready to construct the account or value balance per cycle. Consider cycle t. A value balance relates the value of deliveries to end users (destination) back to the value of inputs and net benefits (source) through the hydrological process. The destination side consists of the value of deliveries to end users during this cycle $p_t' d_t$, plus the discounted value of the deliveries to the future $r y_{t+1}' k_{t+1}$, i.e., the end-of-cycle or *End* stocks. Because of the homogeneity property of the extended functions $\tilde{H}$ and $\tilde{F}$ all end value is attributed, hence, the destination side balances against the source side that consists of the value of the endowments $l_t' w$ plus the value of begin-of-year or *Initial* stocks $y_t' k_t$ plus the value of structural factors $d_t' z_t$. Thus, the basic annual accounting equation reads

$$p_t' d_t + r y_{t+1}' k_{t+1} = l_t' w + y_t' k_t + d_t' z_t , \tag{3.6}$$

and expresses that the value of single current-period's deliveries to end users plus the present value of the deliveries to the future through the end-of-period stock is equal to the value of the endowments, plus the value of stocks, plus the value of structural factors itself at period t. Summing both sides over t yields the total value balance over time, but for most policy decisions the evolution of the annual value balance is of interest. This balance can be presented in tabular form as in table 1.

Table 1. Value balance for cycle t

Destination	Value	Source	value
End users in cycle *t*,	$p_t' d_t$	Natural endowments,	$l_t' w_t$
End stocks cycle *t*,	$r y_{t+1}' k_{t+1}$	Initial stocks cycle *t*,	$y_t' k_t$
		Net benefit cycle *t*.	$d_t' z_t$

Aggregation over all t of the values discounted by r^t on both sides and meanwhile canceling terms appearing on both sides yields the overall account of (3.1), but for most policy decisions the evolution of the value balance per cycle is of interest, which would be an annual account in most applications. Similar as in the previous section, one can zoom into processes within each cycle.

We conclude this section with three remarks. First, the concept of end user should be taken generally. For example, in rain-fed agriculture crop growth is related to evapotranspiration and the latter is a delivery by the hydrological system. Hence, rain-fed agriculture should be regarded as an end-user of evapotranspiration even though there is no market for it and rain-fed agriculture fails to pay the imaginary authority for the amounts delivered meaning the benefits accrue to the farmer or the landowner. Indeed, the main purpose of the approach is to quantify the value of deliveries in situations where markets are absent, for if markets where present and competitive, we might as well retrieve the information from that source.

Second, though we have taken the end use prices as given, the accounting remains valid for any nonlinear, homogeneous welfare function. Third, the shadow prices q_t and m_t do not appear in the value balance. The reason is that after elimination of the variables d_t and $\tilde{k}_t$ from of the optimization problem we obtain the equivalence

$$p' d_t + r\ y_{t+1}' k_{t+1} = p' \tilde{F}(k_t, g_t, z_t) + r\, y_{t+1}' \tilde{H}(k_t, \tilde{F}(k_t, g_t, z_t), g_t, z_t) . \tag{3.7}$$

The function on the right-hand side is homogeneous of degree one, meaning that, by applying Euler's rule, it can be written in terms of the variables k_t, g_t, z_t only. Thus, the shadow prices q_t and m_t do not play any role in the value attribution.

Section 4
Sustainable use

In applications the infinite horizon problem is approximated by a finite horizon solution, say for $t \leq T$ cycles, while imposing $y_{T+1} = 0$, and the sequence of single-period problems (3.5) is solved iteratively backward. The zero value of the final stock k_{T+1} implies that the overall contribution of stocks is undervalued. We propose an alternative approximation for y_{T+1} below, that is also of use in situations where the backward iteration of (3.2) becomes impracticable, for instance if the scale of the hydrological system is simply too large to perform more than a few iterations whereas r relatively close to 1 would require a long horizon. A long horizon also obscures the fundamentals of reproducing capacity of the hydrological processes. In that case, a steady state analysis, with its own drawback, seems the only alternative.

Therefore, we follow an alternative route by evaluating the system at its sustainable or steady state level. Sustainable water use can be defined as the level of use such that the hydrological system is able to reproduce indefinitely. Thus, abstracting from random fluctuations in endowments, aquifers are at the end of every cycle recharged to the level at the beginning. So, sustainability translates into a level of stocks k^* and end use $d^* = F(k^*, w)$ such that $H(k^*, d^*, w) = k^*$. For an autonomous system of hydrological equations, the steady state can be calculated by straight simulation, provided the hydrological system is stable.[7] The simulation should be run until the stocks and end use do no longer change across cycles, i.e., for annual processes the end stock of December 31 coincides with the initial stock of January 1 of the same year. Sustainability also implies that the maximum attainable welfare per cycle is constant across cycles, implying that all shadow prices are constant across cycles, i.e., $y_{t+1} = y_t = y^*$.

A cumbersome way to calculate y^* is to iterate the optimization problem (3.5) for some finite horizon in the steady state k^* starting at some large T and some initial guess for y_{T+1} and then proceeding backward through time until at some $t < T$ the modeler obtains $y_{t+1} \approx y_t$. This procedure requires stability around y^*, which is far less evident than for quantities. In order to avoid this procedure, we implement a different approach in which all shadow prices, including y^*, are computed in a single optimization problem initiated at the steady state k^*. This alternative problem is given by

[7] The hydrological system is stable if there is only one steady state and all dynamic simulations with the autonomous models converge to it, for all stocks in a relevant range. Uniqueness and convergence is seldom an issue in hydrological models of river basins, because stocks of water will not accumulate indefinitely at any location, and surpluses will eventually leave the system. Therefore, the dynamics of stock accumulation satisfy the contraction property and stability holds, e.g., see Ortega and Rheinboldt (1970). However, in some cases convergence may require to reduce the speed of stock adjustment, which essentially amounts to approximating the underlying differential equation with a shorter time step.

$$
\begin{aligned}
& max_{d_t \geq 0, g_t \geq 0, k_{t+1} \geq 0, \tilde{k}_t \geq 0} \; p' d_t \\
& subject \;\; to \\
& \quad d_t \;\; = \tilde{F}(r\, k_{t+1} + (1 - r)\tilde{k}_t, g_t, z_t), \qquad (q_t) \\
& \quad k_{t+1} = \tilde{H}(r\, k_{t+1} + (1 - r)\tilde{k}_t, d_t, g_t, z_t), \qquad (m_t) \\
& \quad g_t \grave{u} \, , = \qquad (l_t) \\
& \quad (1 - r)\tilde{k}_t = \bar{k} \, , \text{ø} \qquad (\;\; _t) \\
& \quad z_t \;\; = 1, \qquad (d_t)
\end{aligned}
\tag{4.1}
$$

for $\bar{k} = (1 - r)k^*$. This optimization problem is trivial in the sense that its physical variables have known values, namely $k_{t+1} = \tilde{k}_t = k^*$ and $d_t = d^*$. The mathematical equivalence between problem (4.1) and problem (3.4) at the steady state is established in Keyzer (2000). To sum up, problem (3.5) can be solved at the steady state in the following three steps.

1. Simulate the hydrological model (without the shadow prices) until the steady state values k^* (and d^*) are reached.
2. Solve problem (4.1) for the shadow prices (powerless river basin authority).
3. Solve problem (3.5) for $k_t = k^*$ and $y_{t+1} = y^*$ yielding the same shadow prices as in step 2.

The third step may seem superfluous, because the second step already yields all the shadow prices we are after. Nonetheless, it provides an essential consistency check, because it should return $k_t = k^*$ and $y_{t+1} = y^*$ when solved for $k_t = k^*$ and $y_{t+1} = y^*$.

This three-step procedure naturally lends itself for implementation in numerical optimization software, and in section 5 we rely on GAMS in this respect[8]. This has the advantage that it does not require evaluation of analytical derivatives, but it does however require the migration of the process-based model from the simulation language in which it was written to an optimization software. Execution time for our application of this three-step procedure on a Pentium IV, 1 GHz, was approximately half a minute.

Alternatively, one could derive shadow prices through numerical approximation in the simulation software itself, as in Hoekstra (2002), by imposing small perturbations on the models parameters, which are located at the right-hand side of the last three constraints in (3.4) for $t = 0$ and $k_0 = k^*$, and measuring the effect on the welfare function in (3.4). Obviously, this avoids the migration to optimization software, but several disadvantages must be noted. First, each parameter should be perturbed in isolation meaning that for each perturbation the simulation software should be run until the change in cumulative, discounted welfare between two subsequent cycles is within the numerical tolerance set by the modeler. It is likely that the latter requires a long time

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[8] GAMS stands for General Algebraic Modeling System. It is widely applied within economics and operation research, because of its flexibility. See also www.gams.com and Brooke et al. (1996).

horizon. Thus, all perturbations must be conducted in sequence and every single perturbation costs computing time. For the AQUA model, the number of initial stock variables and endowment is admittedly modest but the number of structural factors can easily become large. Second, it is far easier to construct the accounts on the basis of a single coherent set of shadow prices than of a series of simulation runs, if only because oversight of any endowment, stock or structural factor creates an accounting gap. Third, the size of the imposed perturbation is not irrelevant. If it is taken too large, higher-order effects will be captured that will create accounting gaps making the valuation meaningless in accounting terms. If it is too low, numerical imprecision may interfere with the marginal valuation. Commercial optimization packages contain scaling devices that automatically deal with these problems.

To illustrate the last point, we implemented the perturbation of monthly precipitation to calculate 'simulated' shadow prices in our application. For monthly increases of 0.1 per cent small second-order effects were obtained that became negligible for shocks of 0.001 per cent. Furthermore, all simulated marginals were within 1 per cent of the marginals obtained in the optimization software, and this error reduces to 0.025 per cent after 100 cycles. Simulations of about 200 cycles were needed to obtain the default accuracy in the optimization package.

As an alternative, we also investigated a computationally more efficient procedure. In this procedure the welfare function in (3.5) instead of (3.4) was used at $t = 0$ and only the first cycle was simulated to obtain d_0 and k_1, where accurately 'simulated' shadow prices for y^* where taken to price the end stocks k_1. It appeared that this approximation performed almost as well as (3.5) with the accuracy after 200 cycles and, as before, second-order effects were avoided if 0.001 per cent shocks were applied. But of course, the accuracy critically depends upon the accuracy of the simulated shadow prices for end stocks, i.e., y^*.

To sum up, for models with a small number of stocks, endowments and structural factors, it is possible, but not trivial, to sidestep the numerical optimization and obtain the shadow prices through simulation. For medium size models, like the AQUA model the simulation approach seems more practical, especially if full accounts are to be produced.

Finally, many finite horizon process-based models are not autonomous because exogenous trends such as population growth, technical change or climate change are incorporated. As an alternative to the 'no value' approximation $y_{T+1} = 0$ mentioned earlier, we note that the steady state valuation could be used to obtain a different approximation for the shadow prices of the end stocks. Population growth and technological change can be treated as stock variables that evolve over time, whereas climate change can be easily modeled as a time-dependent endowment vector w_t. These trend variables are frozen at their $T+1$ levels and, next, the three-step procedure is applied to compute y^*. The latter is taken as the approximation $y_{T+1} = y^*$ in the finite-horizon non-autonomous model, from which the standard backward induction is initialized. This still is an approximation, but one that avoids the zero pricing of stocks in the last period and that takes into account the fundamentals of the reproduction mechanisms.

Section 5
Application to Upper-Zambezi River

We are now in a position to apply the three-step procedure to the AQUA model, a semi-lumped process-based hydrological model of the Upper-Zambezi river basin, developed by Hoekstra (1998). We could benefit from access to this model's original computer code, and were given a file with outcomes of a simulation run towards steady state values for the purpose of comparison. Specifically, the AQUA model describes the entire Zambezi river basin and we apply the valuation method to one of the eight sub-basins being the Upper-Zambezi, which is located in the border region of Zambia and Angola. The area of the sub basin is approx. 19.46 million ha of which 96.1 per cent is grassland and savanna. Rain-fed agriculture covers another 3.5 per cent and irrigated agriculture is limited to 65,000 ha (0.4 per cent). Average rainfall is about 1000 mm per annum.

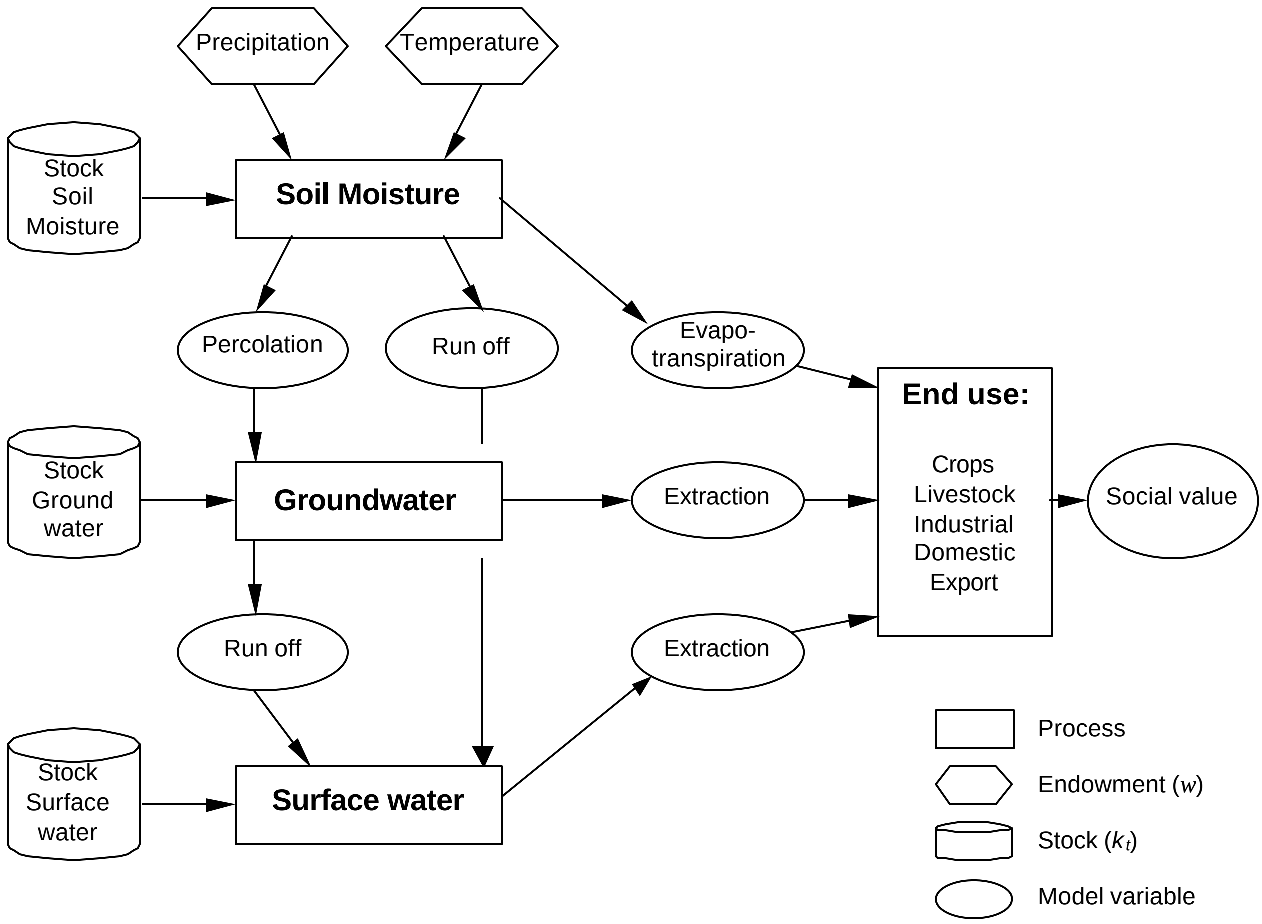

Figure 1 Stylized chart of the main elements and (net) flows of value in AQUA

As shown in flowchart of Figure 1, the sequential processes in AQUA can be grouped into four linked compartments per sub-basin, three describing hydrological processes and the 'fourth' the economy. This flowchart defines the structure of the accounts to be presented in this section and, hence, guides the insertion of accounting rows that separate parts of the recursive model, in the way discussed in Section 2. After discussing the accounts defined by Figure 1, we will zoom in on the equation of potential evapotranspiration. The endowments and the stocks appear on the upper left corner of this figure. Structural factors are not shown, but they would enter the processes like endowments. Simulation fully determines the sum of discounted values over the infinite horizon of (3.4) and the objective only serves the purposes of valuation, and multipliers effectively determine the change in this objective under a small perturbation in the endowments, stocks, and structural factors.

In terms of section 3 the three compartments on the left represent function H. The model equations are presented in Appendix B in its original notation and translated into the functions H and F in Appendix C. Each compartment is based upon a water balance. The soil moisture compartment distinguishes three land-cover types (grassland-savannah, irrigated agriculture, rain-fed agriculture), and for each type, monthly precipitation is distributed by Thornthwaite formula over soil moisture, evapotranspiration and, if in excess, distributed in fixed proportions over direct runoff and percolation. Surface and groundwater are each governed by the storage principle. In the full model of the Zambezi basin, surface water flows out to the adjacent downstream cell, but in our lumped one-cell application it leaves the system, and is treated as part of end use in order to introduce some rivalry.

Economic activity constitutes the compartment on the right and corresponds to the function F. As mentioned above, to keep the focus on the hydrological process, we truncate this part of the AQUA model and replace it by a valuation of all end use at a given price, and a discount factor of .95.[9] Each end use category obtains its water in fixed proportions from either ground and surface water and induces a return flow. Rain-fed and irrigated agriculture are treated as end users of evapotranspiration. Whereas rain-fed agriculture fully depends on evapotranspiration, irrigated agriculture additionally receives irrigation water applied. Finally, the surface water leaves the Upper-Zambezi basin at a given price, and introduces a certain degree of competition for water between the Upper-Zambezi and its downstream users.

A multi-cell application could proceed by conducting a valuation of the most downstream cell, for a given inflow from the next higher cell. This yields a stock valuation for this inflow that can be used to value the surface flow in that cell, and so on, until the Upper-Zambezi. Hence, the relation between the cells is very similar to the dynamic linkages between periods.

[9] The updated version of AQUA in Seyam et al. (2003) relates the value added of each sector's water use to the market price of the final product through the inverse demand function.

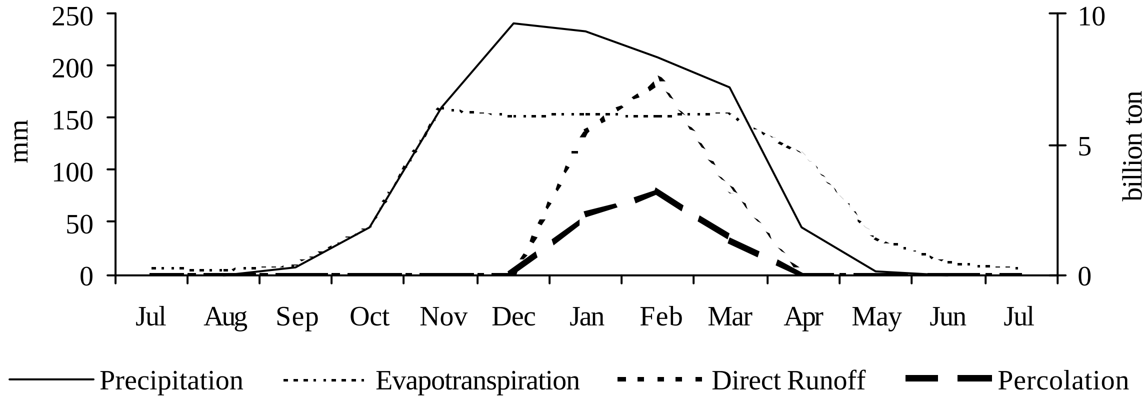

Figure 2. Annual cycle of precipitation and evapotranspiration (left axis), direct runoff and percolation (right axis) in the Upper-Zambezi basin.

5.1 Steady state

The valuation is conducted for a steady-state scenario, with all quantities determined by AQUA. We briefly sketch this base run scenario. There are two seasons, a wet season (December until April) with high monthly average temperatures and a dry season with cooler temperatures. On grassland and savannah, the topsoil becomes fully saturated during the wet season, resulting in three months of direct runoff and percolation. During these months, the reservoirs of surface water and groundwater are recharged. After the wet season the soil moisture vaporizes and by the end of the dry season it has almost dried out. For the other two land-cover types, irrigated and rain-fed land, the precipitation is insufficient to generate any recharge of the reservoirs. Figure 2 illustrates this annual cycle in steady state.

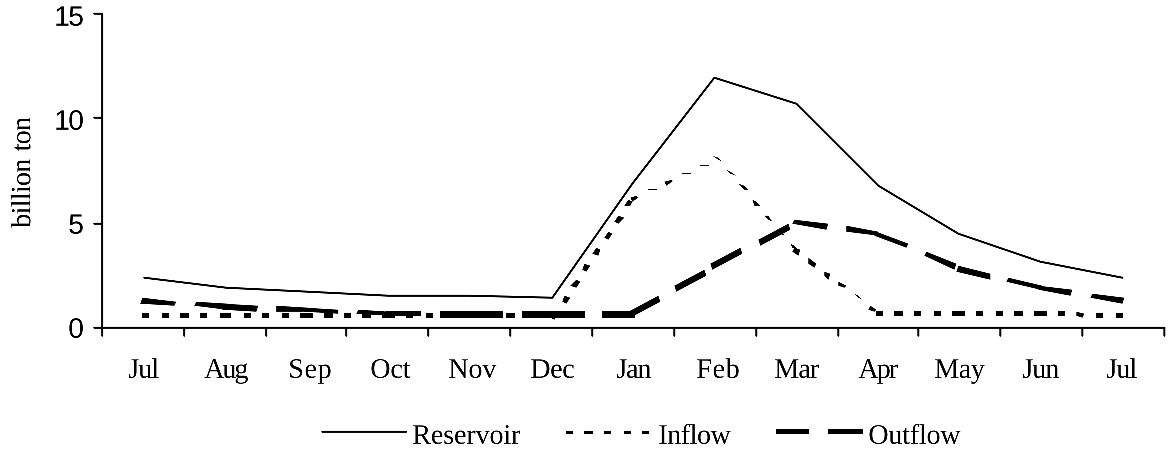

Figure 3. Annual cycle of the surface water reservoir and the in- and outflow.

The groundwater reservoir is fully recharged during the wet season. The natural, monthly outflow is a constant fraction of the groundwater stock in that month, i.e., the storage principle. The groundwater reservoir is huge (473 billion ton) and the runoff fraction is rather small, which causes an almost flat pattern in monthly runoff and the reservoir level during each year. Of course, in steady state, each year's recharge equals annual runoff. Surface water is also governed by the storage principle but its flow variations between months are more pronounced since the surface runoff during the wet season leaves the surface water within a few months. The steady state surface water cycle is illustrated in Figure 3.

Annual total human water use in the Upper-Zambezi is only a small fraction (6 per cent) of natural flows, of which 78 per cent accrues to irrigated agriculture, followed by domestic and industrial water use of 11 per cent and 8 per cent, respectively, and the remaining 3 per cent are for drinking requirements of livestock. Following AQUA, the water use per sector is assumed constant across the year, except for irrigated agriculture, where application is constant over eight months, starting in April.

5.2 Valuation

Having analyzed the AQUA model, the next step is to transfer the model from its original simulation software to an optimization package that allows for calculation of multipliers, where each statement in the computer code becomes one of the equations in (3.4). We refer to Appendix D for details on the modifications to the AQUA model needed to perform the valuation and E for a more detailed discussion of the actual migration. Equations to price the stocks and endowments as well as the accounting rows and structural factors for the compartments were introduced. The latter expands the AQUA model but does not cause changes in the physical solution. The transfer to GAMS passes the major test of replicating AQUA's base run. Next, the optimizations (4.1) followed by (3.5) are run in order to extract the multipliers for the accounts. Since the quantities are fixed and known in advance these optimization problems are evaluated for these quantities. As discussed in section 3, running (4.1) is sufficient, but we also ran (3.5) in order to test the implementation of (3.3) and our implementation passes this test. All results are presented as accounts in tabular form of Table 1 of section 3, which correspond to (3.5) in the steady state. The accounts are recursively linked, starting from the end users on the right side in Figure 1, and moving to the stocks and endowments.

In Table 2, the deliveries from the three hydrological compartments to the end users appear to have almost equal value. Throughout the Zambezi-river basin, agriculture contributes 20 per cent to GNP and our value balance is calibrated to reproduce this percentage of the value generated within the Upper-Zambezi. Furthermore, also the value accruing to downstream use was calibrated to the level shown reflecting that the marginal contribution of water that passes to downstream is probably low for several reasons. First, much of downstream economic activity is

Table 2. Annual value account of the End users

Destination	Value	Source	Value
End users		Surface water	
Domestic use & Industry	44.724	Runoff	10.632
Savannah & grassland	35.596	Extraction	0.044
Rain-fed land	38.848	Groundwater	
Irrigation land	43.894	Extraction (*Net*)	-0.001
Downstream	10.632	Soil moisture	
		Evapotranspiration	118.338
		Other inputs end users	44.681
Total	173.694	Total	173.694

some distance away from the riverbed or occurs in tributaries of the Zambezi and, therefore, these activities do not contribute any marginal value to the water passed on by the Upper-Zambezi. Second, the runoff from the Upper-Zambezi flows into the thinly populated wetlands of the Barotse plain, which has a somewhat dryer and hotter climate. This means that in this area not much value through economic activities is generated and, more important, the marginal productivity generated further downstream is reduced by substantial losses due to evaporation. As a consequence, the marginal value of runoff from the Upper-Zambezi will be low. Yet, we expect that roughly 6 per cent of the annual value in the value balance by economic activities is generated downstream. Obviously, this is only an assumption it would be straightforward to apply a different price.

Prices and values in the tables below are measured in a hypothetical currency. This does not distort the figures, since, as can be seen from optimization problem (3.4), only changes in price ratios and the discounting factor, affect the value distribution. The total value of deliveries, shown on the right hand side of the table, originates from the three hydrological compartments in the model and from the non-water inputs in the activities of Domestic use and Industry. In this model these hardly depend on water, as can be expected in a region without major cities and industries. Almost all downstream use comes from Surface water. The main in-basin use is agricultural, and derived from Soil moisture. The negative value for Groundwater extraction is the net annual result. Replenishment of groundwater is valued higher than extraction. This suggests that groundwater mining would be detrimental in this region.

Next, separate annual accounts for each hydrological compartment are given in Table 3-5. Note that, if compared with Table 2, that the values of inflows in the latter table appear in the accounts per compartment as the value of deliveries, which illustrates the transfer of value through internal deliveries. Furthermore, for each compartment the End stock in a given year, being next year's Initial stock, now appears separately. Since we consider a steady state, the End stock value is by construction equal to the Initial stock value multiplied by the discount factor. As can be seen from Table 3 and 4, the value of stocks for Surface water is quite different than for Groundwater. For Surface water the essential information is that the importance of carryovers to future years is limited, as the End stock only comprises a minor fraction of total value in this compartment. This is because channels and rivers quickly pass through the wet season's surface runoff and the subsurface runoff. Therefore, carryovers to next year are limited as there are no large dams to control surface water, which seems a plausible finding for the Upper Zambezi. For Groundwater we can infer quite the opposite. The stock value is remarkable high and reflects the importance of groundwater in the hydrological system. Yet, this value is not comparable to say the extraction value of ores in a mine. Rather it is comparable to the value of land as a store of fertility and reflects the discounted value of the contribution to future production.

There are some other interesting findings. In the Surface water compartment the direct runoff is the main source of water, more than twice as much as the Runoff from groundwater. However, Direct runoff is concentrated in the wet season, while groundwater is the important source during the dry season as water stress increases. This is also reflected in the multiplier that is almost 60 per cent higher for groundwater runoff that underlies the reported values. With respect to the Groundwater compartment Percolation is the only water source. Since the basin has no runoff in deep soil layers the main outlet is Runoff to surface water. Note that since net extraction of groundwater is negative it has a negative value, but effectively it has become an inflow (source) rather than an outflow.

Table 3. Annual value and volume account for the surface water compartment

Destination	Volume*	Value	Source	Volume	Value
Downstream basins	22.946	10.632	Direct runoff	16.101	6.354
Extraction in Upper-Zambezi (*Net*)	0.059	0.044	Runoff from groundwater	6.904	4.308
End stock	1.415	0.266	Initial stock	1.415	0.280
Total	24.420	10.942	Total	24.420	10.942

*Volumes in billion ton

Table 4. Annual value and volume account for the Groundwater compartment

Destination	Volume*	Value	Source	Volume	Value
Runoff to surface water	6.904	4.308	Percolation	6.900	0.970
Extraction	-0.004	-0.001			
End stock	473.315	63.413	Initial stock	473.315	66.750
Total	480.216	67.720	Total	480.216	67.720

*Volumes in billion ton

The most complex processes within AQUA take place in the Soil moisture compartment. The values of the flow destinations already appeared in previous tables. Here we present a breakdown by land cover type. The largest value accrues to irrigated land despite the fact that it has the smallest area. The importance of irrigated land is also reflected in the value attributed to one hectare, which is about 120 times the value of rain-fed land. This illustrates how the valuation can be used in the cost-benefit analysis of irrigation development, albeit that here the cost side is not

Table 5. Annual value account for the Soil Moisture compartment by land cover type.

Destination	Grassland & Savannah	Rainfed Land	Irrigated land	Total value
Evapotranspiration	35.596	38.848	43.894	118.338
Percolation to ground water	0.970			0.970
Direct runoff to surface water	6.354			6.354
End stock	3.575	1.550	1.686	6.811
Total	46.495	40.398	45.580	132.473
Source				
Precipitation	48.068	37.641	41.723	127.432
Potential evapotranspiration	-5.336	1.125	2.082	-2.129
Initial stock	3.763	1.632	1.775	7.170
Total	46.495	40.398	45.580	132.473

accounted for. Before discussing the source side of the account, some additional explanation of the entries is in order.

So far, we have not distinguished explicitly between the endowments and the structural factors as contributors to value in the source part of the accounts. In fact, it is possible to zoom in much deeper within the production process, both time wise, say, to present accounts on a monthly base, and technologically on the ‘production’ process itself, identifying the contribution of the various structural factors, rather than lumping these together into aggregate totals. For this, we can in between any two stages of the (recursive) process, distinguish the variables that enter and those that leave and treat them as sources and destinations. As long as the homogeneity property was adequately imposed through structural factors, the resulting accounts will balance. For instance, in the soil moisture component the potential evapotranspiration is calculated on the basis of the temperature,[10] and used jointly with precipitation and the current stock of moisture to calculate a new stock as well as actual evapotranspiration and direct runoff and percolation. In Table 5 Potential evapotranspiration appeared as an entry to measure the effect of temperature. It appears that there is a negative value for Grassland and Savannah, and this calls for further explanation, which means that we must zoom in deeper into the value flows inside this compartment.

At first glance, a possible (but flawed) reason for the negative value might be that the potential evapotranspiration depends on temperature via a convex function through the origin.[11] Recall the explanation of Euler’s rule and accounting in section 2. The multiplier of temperature multiplied by temperature itself now exceeds the value of potential evapotranspiration, the output of this process. The gap is filled by the structural factor, which is negative for a convex function through the origin. However, as long as the total value of the process is positive, this should result in a positive value for potential evapotranspiration itself, but it appears in the table that this value is negative for Grassland & Savannah. Hence, the explanation thus far fails meaning the negative sign is not attributable to the convexity. Rather it seems that on balance higher evapotranspiration

[10] AQUA uses a modified Thornthwaite equation as stated in (A-2) of Appendix A.

[11] The partial derivative of (A-2) with respect to the structural factor z is $f_z'(T) = (1-a)T^a z^{-a}$ and, hence, negative for $a = 2.36$.

in the AQUA model contribute negatively to economic value, or in plain terms, that a temperature rise as might be expected under climate change would be detrimental. This illustrates a typical feature in ecological modeling as opposed to economic modeling. A profit maximizing firm would not apply more inputs than could contribute to profits. But (outside) temperature is dictated by nature and excess heat cannot be disposed of at zero cost. Thus, the valuation method makes it possible to quantify the natural advantages and handicaps of a particular region, as expressed by the model under study.

We are now ready to present the aggregated annual account of the Upper-Zambezi basin in Table 6, i.e., Table 1 of section 3 for this application. Thus, Table 6 gives on the left-hand side the total value of the demand and the discounted value of the End stocks $p'd_t + r\, y_{t+1}'k_{t+1}$ and on the right-hand side the total value for endowments, Initial stocks and the net benefits of the hydrological processes: $m_t' w + y_t' k_t + n_t' z_t$. Summing the earlier accounts per compartment and canceling terms appearing on both sides can obtain this account. Alternatively, this table can also be obtained by grouping all the processes together into one large compartment, i.e., no zooming in, but if accounts per compartment are already derived as in our case then transparency favors the summing of accounts. The stock value of the aquifer is the largest among the stocks. Since we only account for the valuation of the Initial and End stock, i.e., priced at December 31 in the wet season, no value is given for the dry season. The groundwater stock is an important asset in the basin covering almost 30 per cent of the value. On the source side, Precipitation is the only endowment, Hydrology stands for the total value of the structural factors and the other inputs are non-water inputs.

Table 6. Consolidated accounts (in value terms)

Destination	Wet	Dry	Annual	Source	Wet	Dry	Annual
Upper Zambezi				Precipitation	127.361	0.071	127.432
Evapotranspiration	116.925	1.413	118.338	Hydrology	-3.470	1.341	-2.129
Other sectors	5.756	38.968	44.724	Other inputs end users	18.600	26.081	44.681
Downstream	1.368	9.264	10.632				
End stocks	70.490	0.000	70.490	Initial stocks	74.200	0.000	74.200
Total	194.539	49.645	244.184	Total	216.691	27.493	244.184

Section 6
Concluding remarks

The paper reports on the application of a capital-based valuation method to the hydrological component of the AQUA model, for the Upper Zambezi. Since the value of end use is only represented by means of given and somewhat hypothetical prices, the numerical findings are of illustrative value only. The primary aim has been to show that the approach technically works: it can accommodate a given process-based model without any modification. For this model it can determine steady state prices over an infinite time horizon, and produce complete accounts for all elements of interest, including structural factors that reflect the properties of the process themselves, in addition to the natural inflows of water and the initial stocks. In these accounts, the value of deliveries by the hydrological system is fully attributed to, and therefore balances with, the value of resources, including the physical characteristics that are reflected in the shape of the functions in the model. It also naturally suggests the use of a Lagrange multiplier method to compute the shadow prices for valuation, while avoiding the need to impose an immensely large number of small perturbations on the model, or to calculate all derivatives analytically. By contrast, in the literature one finds marginal valuations that only partially exhaust the total value, simply because the structural factors are overlooked.

The valuation considered so far assumes that the river basin authority only monitors the situation, by assessing the welfare implications of essentially virtual perturbations. This is computationally powerful, because the time path of physical variables can be computed through simulation. The natural next step would be to allow the authority to implement a sequence of such perturbations. Recall that the shadow prices are the gradients of the welfare objective with respect to the fixed right hand sides of the program. Hence, they can be used as the directional search step in a gradient algorithm, that changes (some of) these right-hand side values in the direction of the gradient, i.e. increases the level of the constants with positive shadow prices and reduces that of those with negative right hand sides, to the extent that this is technically possible. For example, it would be meaningless to change the unit value of the structural constants but one might envisage changing, at a cost, the shape of, say, the river by reducing a particular meanders. For small enough adjustments, each iteration of this extended algorithm will improve the welfare function until some local optimum is reached. Of course, as with any optimization model, one point of concern is that the sequence of optimal solutions does not drift outside the region of parameter values that can be validated through calibration of the original process-based model.

The tool could also be used to assess the consequences of larger, scenario driven changes, such as the construction of a major dam or the pumping of water to another watershed. The hydrological model could be used to simulate the physical effects, and the optimization would then indicate on the one hand how the change affects welfare, and, more importantly, are it affects relative scarcities in the system.

Future extensions of the method could proceed in several other directions. One would be to apply it to less stylized process-based models, for example to account for variability in climatic conditions. The use of Bellman’s principle applies fully in this case as well. Another extension would be to turn to a spatially explicit setting that is to describe the flows between the grid cells of a geographical map. In such a GIS-based setting, even the computational technique applied to optimization problem (4.1) becomes intractable. In Keyzer (2002) an improved algorithm is proposed that can effectively deal with this case by further exploiting the recursive structure of the process-based model. This algorithm can also change the right hand side values in a gradient direction, at a first stage to calibrate the model, with the planner’s objective replaced by a measure of fit to the data, and at a second stage, to orient policy interventions on the basis of their contribution to the regular welfare objective.

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Appendix A: Numerical examples of shadow prices and value accounts

In this appendix the theoretical discussion of section 2 is illustrated by the means of a simple numerical example. The interpretation of shadow prices is also addressed.

The hypothetical example

Consider the hypothetical case of a farmer who produces a single crop d by means of the cumulative amount x of irrigation water and whose production function is given by the concave function $f(x) = x^{0.75}$, which has the analytical derivative $f'_x(x) = 0.75x^{-0.25}$. Furthermore, we suppose that the farmer can sell its crop against the price $p = 16$ (per unit). The irrigation authority charges $w = 3$ and imposes the upper bound $\bar{x} = 224$ upon the amount of irrigation water applied. Maximization problem (2.3) becomes

$$\begin{aligned} &\max_x 16x^{0.75} - 3x \\ &\text{subject to} \\ &x = 224. \qquad (l) \end{aligned} \tag{A-1}$$

Simple substitution of $x = 224$ into the objective function yields a profit 254.416, but it cannot not yield the shadow price l. Simulation of process-based models can be considered as recursive substitution in a more complex setting that makes it possible to calculate the overall welfare objective. The farmer's financial account is given in Table A-1, which is stated in volumes, prices and values.

Table A-1. The firm's volume and value balance including prices

Destination	Volume	Price	Value	Source	Volume	Price	Value
Delivery *d*	224.000*	4.136	926.416	Input *x*	224.000	3.000	672.000
				Profit	idem	1.136	254.416
Total	224.000	4.136	926.416	Total		4.136	926.416

* 224.000 units of water produce 57.901 units of crop output

On the interpretation of shadow prices

Implementation of (A-1) in the numerical optimization software package GAMS yields the same profit as in Table A-1 and the shadow price $l = 0.102$. This price indicates the linear approximation of the change in the farmer's profit to small changes in x. For the nonlinear profit function the approximation becomes less accurate the larger the change in $\bar{x}$ is, as illustrated by Table A-2.

Table A-2. The change in profit and its linear approximation $\lambda\,(x-\overline{x})$ at $\overline{x}=224$.

Volume x	Profit at x	Change	$\lambda\,(x-\overline{x})$	Difference per unit of x
225	254.516	0.100	0.102	0.002
226	254.612	0.196	0.204	0.004
229	254.882	0.466	0.510	0.009

The relation between the profit level, the curvature of the function and the derivative of the profit function, is illustrated in Figure A-1. The shadow price λ depends upon the curvature where its value is computed. A shadow price of zero indicates that the profit function attains its maximum at the corresponding value of x, which is at $\overline{x}=256$ (the vertical line in Figure A-1).

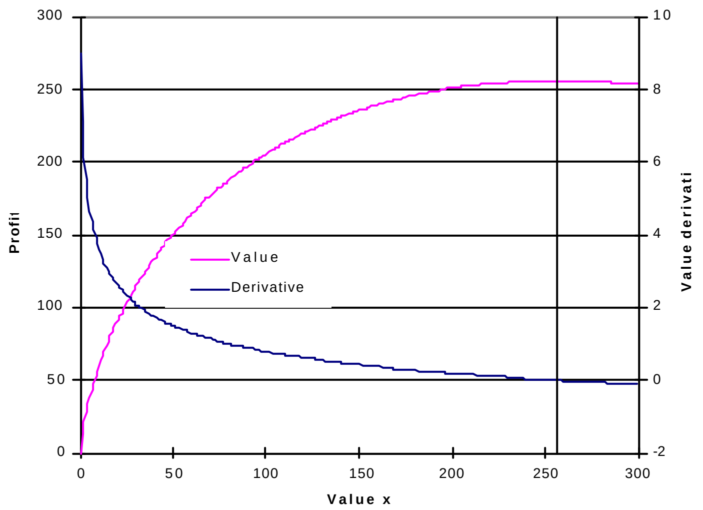

Figure A-1 The profit function and its derivative

In the case of irrigation water the farmer would never voluntarily purchase more than 256 units of irrigation water. However, if the farmer is forced to apply an excessive amount $\overline{x}$ then negative values of λ indicate that the farmer's profit would increase if the level $\overline{x}$ could be lowered. Note that the shadow price also depends upon the market price w. This follows from equation (2.4), which reads as

$$pf_x'(\overline{x})=12\overline{x}^{-0.25}=w+\lambda=3.102\,. \qquad \text{(A-2)}$$

Under the inequality $x\le 224$ the farmer would still want to purchase the amount $x=224$ as long as the price of irrigation water does not exceed 3.102. Only if the authority would charge more the farmer prefers to reduce its input below 224. The amount $\lambda\overline{x}=22.848$ should be regarded as an implicit *subsidy* from the irrigation authority to the farmer, because the productivity of irrigation water received by the farmer is higher than paid for through the price w. Of course, a higher charge involves an income transfer from the farmer to the irrigation authority. Note that the subsidy is especially relevant for natural resources such as precipitation for which the absence of markets effectively means a market price w of zero and a l that fully measures the marginal productivity of the resource. This is one reason why land has economic value; a farmer is willing to buy land or pay rent to the landowner to gather the benefits from nature.

Finally, we address the issue of how to approximate shadow prices without application of optimization software and what the pros and cons of such approach are. The shadow price l can be approximated as follows: Compute the change in profit if $x = 224$ increases to, say, $x = 225$, which is similar as in the third column of Table A-2, and then divide the computed change in profit by the increase in x, which yields an approximated shadow price of 0.100. Table A-3 illustrates how such approximation depends upon the size of the change in $\bar{x}$ and this table also includes negative changes. This table shows that the approximated shadow price only coincides with $l = 0.102$ for very small changes from $\bar{x} = 224$ and that the quality of the approximation deteriorates the larger the change. This effect is due to the nonlinear curvature of the profit function, where second and higher order effects come into play that determine the curvature. Since the appropriate size of the small change is a priori not known, this is a major disadvantage of the approximation approach. However, there is a simple test to determine the quality of each approximated shadow price: The approximated shadow price corresponding to a small increase should be (almost) equal to the approximated shadow price of a decrease of equal magnitude. Applied to Table A-3 and imposing a numerical tolerance of 0.001, this test implies that the change should not be larger than 0.25, i.e. less than 0.1 per cent of $\bar{x} = 224$. It is important to note that too small changes might result in numerical problems, because both the numerator and denominator, i.e. the change in the objective and x, respectively, in the calculation become small.

Table A-3. Approximated shadow prices

Change in $\bar{x}$	0.100	0.250	0.500	1.000	3.000	5.000
Simulated price	0.102	0.101	0.101	0.099	0.095	0.086
Change in $\bar{x}$	-0.100	-0.250	-0.500	-1.000	-3.000	-5.000
Simulated price	0.102	0.102	0.103	0.104	0.107	0.112

Valuing resources and processes

As discussed in section 2, accounts as Table A-1 can be easily extended to include additional information, such as the implicit *subsidy* $lx = 22.848$ and the net profit that can be attributed to 'technology', i.e., the shape of the function *f*. Here the technology means the value generating capacity of the natural physical processes that govern plant growth. This can be measured by introducing the structural factor $z = 1$ and redefining the production function as

$$\tilde{f}(x,z) = zf(x/z) = z(x/z)^{0.75} = z^{0.25}x^{0.75}. \qquad \text{(A-2)}$$

It is easy to see that doubling both x and z doubles the function value of $\tilde{f}$ and, similar, for any (positive) multiplication of these variables. Thus, the function $\tilde{f}$ is homogeneous of degree one and Euler's rule applies, meaning

$$\tilde{f}(x,z) = \tilde{f}'_x(x,z)x + \tilde{f}'_z(x,z)\ z = (0.75z^{0.25}x^{-0.25})x + (0.25z^{-0.75}x^{0.75})z = z^{0.25}x^{0.75}. \qquad \text{(A-3)}$$

The extended profit maximization problem becomes

$$\begin{aligned} &\max_{x,z} 16z^{0.25}x^{0.75} - 3x \\ &\text{subject to} \\ &\quad x = 224, \qquad (l) \\ &\quad z = 1, \qquad (d) \end{aligned} \qquad \text{(A-4)}$$

and numerical implementation into GAMS yields the same results as before and $d = 231.604$. The farmer's account is given in Table A-4 and the main difference with Table A-1 is that the net profit is now partitioned into subsidy and the value generating capacity of the natural processes. Obviously, Table A-4 contains more information than Table A-1. For policy makers the ‘subsidy’ to the farmer is of importance, but also the consequences to the farmer's income if the bound $\bar{x} = 224$ is lowered. Note also that Euler's rule applied to the revenue function $16\tilde{f}(x,z)$, where $z = 1$, and (2.7) mean that

$$16\tilde{f}(x,1) = 16\tilde{f}'_x(x,1)x + 16\tilde{f}'_z(x,1)\ z = (3 + l)x + d. \qquad \text{(A-5)}$$

Hence, the right-hand side of Table A-4 balances against the left-hand side, which is easily verified from this table. As mentioned in section 2, this property holds in general.

Table A-4. The firm's volume and value balance including prices

Destination	Volumes	Prices	Values	Source	Volumes	Prices	Values
Delivery *d*	224.000	4.136	926.416	Purchases	224.000	3.000	672.000
				Subsidy	idem	0.102	22.812
				Natural processes	idem	1.034	231.604
Total	224.000	4.136	926.416	Total	224.000	4.136	926.416

Valuation in sequential processes

Hydrological systems typically consist of a sequence of processes and we therefore consider the simple case where the previous production function corresponds to two of such processes. In doing so, we additionally suppose that one process is described by a nonconvex technology.[12] Suppose $f(x) = g(h(x))$ is given by $h(x) = x^{1.5}$ and $g(q) = q^{0.5}$. Then $f(x) = x^{0.75}$ as before, the function g is concave whereas the function h is convex. In order to implement maximization problem (2.12) two adjustments are essential. First, each production function is redefined by including its own structural factor:

$$\tilde{g}(h, z_g) = z_g (q / z_g)^{0.5} = z_g^{0.5} q^{0.5}, \quad z_g = 1, \tag{A-6}$$

and

$$\tilde{h}(x, z_h) = z_h (x / z_h)^{1.5} = z_h^{-0.5} x^{1.5}, \quad z_h = 1. \tag{A-7}$$

The two structural factors make it possible to value the contribution of each process separately. Second, to value the delivery from upstream to downstream it is necessary to introduce the accounting variable y and the accounting equation $q = y$, where y also denotes the delivery from upstream. The shadow price of this equation, denoted as f, measures the marginal contribution of a small increase in q or, equivalently, in y. Then maximization problem (2.12) becomes

$$\begin{array}{llllll} \max_{x,y,q,z_g,z_h} 16d - 3x & & & & & \\ \text{subject to} & & & & & \text{(A-8)} \\ d = z_g^{0.5} q^{0.5}, & z_g = 1, \ (d_g) & q = y, & (f) & \textit{Downstream} & \\ y = z_h^{-0.5} x^{1.5}, & z_h = 1, \ (d_h) & x = 224, & (l) & \textit{Upstream} & \end{array}$$

The numerical implementation in GAMS yields, beside the familiar numbers, $d_g = 463.208$, $d_h = -231.604$, $q = y = 3352.525$ and $f = 0.138$. Two results stand out. First, the upstream process generates a negative contribution. The reason is that the underlying technology is nonconvex, because it is described by a convex production function. Second, the marginal contribution of upstream deliveries to downstream is worth 0.138 per unit. This price is also equal to the marginal producer costs for the upstream process and, as such, it can be regarded as imitating the market price on markets with perfect competition. So, even in the absence of markets for deliveries between two hydrological processes the valuation method is able to reveal underlying market prices. For each process we can set up a separate volume and value account as presented in Table A-6A and A-6B. Aggregation would yield a table identical to Table A-4, after canceling terms appearing on both sides.

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[12] In economics a production *technology* is called convex if it can be described by a concave production *function*. Every production function that is not concave is said to correspond to a nonconvex technology, see e.g. Mas Colell et al. (1995). Note that a production function that is convex is a special case of a nonconvex technology, because such technologies also include production functions that are partly concave and convex as e.g. $f(x) = (1-x)^{1/3} - 1$.

Table A-5A. The volume and value balance (including prices) for the upstream process

Destination	Volumes	Prices	Values	Source	Volumes	Prices	Values
Delivery *y*	224.000	2.068*	463.208	Purchases	224.000	3.000	672.000
				Subsidy	idem	0.102	22.812
				Natural processes	idem	-1.0339	-231.604
Total	224.000	2.062	463.208	Total	224.000	2.062	463.208

* 224.000 units of water produce 3352.525 units of intermediate product.
** Internal price determined by optimization problem (A-8) is 0.138 per unit of intermediate product.

Table A-5B. The volume and value balance (including prices) for the downstream process

Destination	Volumes	Prices	Values	Source	Volumes	Prices	Values
Delivery *d*	3352.525	0.276	926.416	Purchases from Upstr.	3352.525	0.138**	463.208
				Natural processes	1.000	0.138	463.208
Total	3352.525	0.276	926.416	Total	3352.525	0.276	926.416

* 3352.525 units of intermediate product produce 57.901 units of crop output.
** Internal price determined by optimization problem (A-8).

We conclude this appendix with several remarks. First, the numerical example demonstrates how adding variables and equations can retrieve information about values from simulation models. However, adding these enlarges the model and, therefore, only the important and relevant processes should be separated, which requires incorporation of a priori insight in the natural processes by the expert. If in the numerical example the policy maker is not interested in detailed insight, then the accounting variable and equation can be deleted from (A-8) and both structural factors z_g, z_h can be replaced by a single structural factor z, resulting in Table A-4. We note that the method is flexible in the desired level of detail. Second, the method for approximated shadow prices can also be applied in (A-8). Correct implementation requires that the right-hand sides of the equations $z_g = 1, z_h = 1, q = y, x = 224$ should be perturbed in isolation. So, retrieving the four shadow prices requires running the 'simulation' four times in a row and for large simulation models such an exercise becomes quite time consuming. Third, in case the downstream process has its source in the confluence of two tributaries then the right-hand side of the accounting row in (A-8) would consists of the sum of inflows from these tributaries.

Appendix B: The process-based model AQUA

In this appendix the process-based model AQUA, as introduced in Hoekstra (1998), is discussed in its conventional hydrological notation. The relation with the abstract notation of section 3 is postponed to the next appendix and Appendix D deals with the modifications necessary for shadow pricing and accounting. Appendix E deals with the software migration of AQUA. This appendix is concluded with a brief discussion on the interpretation of prices for deliveries to end users.

One remark is in place. As for most process-based hydrological models, the evolution of stock variables in the AQUA model is treated as a system of differential equations and in order to obtain numerical solutions this system needs to be embedded, calibrated and simulated in some software package. In most of such packages the modeler can set the numerical method to approximate the variables' time paths, including the Euler or first Runga-Kutta (RK-1) method and the second Runga-Kutta (RK-2) method. In Hoekstra (1998) the RK-2 method with a two-week time step is applied to calibrate and simulate the AQUA model. The valuation method of the AQUA model is implemented under the RK-1 method with monthly time steps and in this appendix this model is presented as a system of difference equations resulting after the application of the RK-1 method. Both the RK-1 and RK-2 method are discussed in Appendix F and there it is shown that the valuation method can handle either one of these methods, but for reasons discussed in Appendix E the RK-1 method was preferred.

The AQUA model

The AQUA model contains eight cells describing the entire Zambezi-River basin and our application focuses upon the single cell describing the Upper-Zambezi, which is the area of tributaries of the confluence near Lukulu in Northwest Zambia. Within this cell there is no further spatial differentiation and a lumped model represents the hydrological processes. This model is modular in its setup and distinguishes four compartments: three describing hydrological processes in topsoil, groundwater reservoir and surface water, and a fourth describing the economy. Hydrological cycles are annual cycles and for convenience we identify each cycle with a calendar year. We refer to Figure 1 in section 5 for an illustration of the modular setup of the AQUA model and start our description of this model with describing the soil moisture compartment.

The soil moisture compartment

The soil moisture compartment represents the top of the recursive structure that feeds the other compartments and it does not receive input from the other compartments. Within the soil moisture compartment three different land cover types are considered; Savannah and Grassland, Rain-fed agricultural land and Irrigated agricultural land. Months within one year are denoted as $m = 1, \ldots, 12$, where the first month represents January. It is convenient to impose the convention that $m-1$ means the previous month, which is last year's December in case $m-1=0$. Within this compartment monthly precipitation first contributes to monthly evapotranspiration and the moist held by the topsoil and, then, only if it is in abundance, it feeds groundwater (percolation) and surface water (direct runoff).

The first two equations represent the natural resources precipitation and temperature, where $P_{lct}(m)$ denotes monthly precipitation (mm) per land cover type (lct) and $T_{lct}(m)$ the monthly average temperature (C^o) per cover type. This leads to the following equations

$$P_{lct}(m) = \overline{P}_{lct}(m), \tag{B-1a}$$

$$T_{lct}(m) = \overline{T}_{lct}(m), \tag{B-1b}$$

where each upper bar denotes exogenous parameters. In AQUA precipitation differs per land cover type, while temperature does not.

The level of soil moisture (mm) per cover type stored by the topsoil at the *end* of the month, denoted as $K_{SM,lct}(m)$, is a function of potential evaporation (mm), denoted as $E_{Pot,lct}(m)$, based on Thornthwaite (1948), and given by

$$E_{Pot,lct}(m) = c_{lct}(m) \cdot T_{lct}(m)^a, \tag{B-2}$$

where $a \approx 2.36$ and $c_{lct}(m)$ are constants,[13] and

$$K_{SM,lct}(m) = \begin{cases} K_{SM,lct}(m-1)\, e^{\left(P_{lct}(m) - E_{Pot,lct}(m)\right)/\overline{K}_{SM,lct}}, & \text{if } P_{lct}(m) < E_{pot,lct}(m), \\ \min\left\{K_{SM,lct}(m-1) + P_{lct}(m) - E_{Pot,lct}(m), \overline{K}_{SM,lct}\right\}, & \text{if } P_{lct}(m) \geq E_{pot,lct}(m), \end{cases} \tag{B-3}$$

where $\overline{K}_{SM,lct}$ is the land cover specific soil moisture maximum capacity, which is an exogenous parameter.

The next block of equations govern the monthly actual evaporation (mm) per cover type, the monthly cumulative percolation (billion ton) into groundwater and monthly cumulative direct runoff (billion ton) to surface water. These three variables are denoted by $E_{act,lct}(m)$, $DR_{lct}(m)$ and $P_{perc,lct}(m)$, respectively. The last two variables depend upon net precipitation (mm) per cover type, which can be regarded as an auxiliary variable representing the abundance in moist

[13] In Thornthwaite (1948) equation (B-2) consists of two separate equations given by

$$I_{Heat,lct} = \sum_{m=1}^{12} 0.2 T_{lct}(m)^{1.514} \quad \text{and} \quad E_{Pot,lct}(m) = q_{lct}(m) \cdot \left(T_{lct}(m)/I_{Heat,lct}\right)^{a(I_{Heat,lct})},$$

where $I_{Heat,lct}$ is a heat index and a is a function of this index, e.g., see Hoekstra (1998) for details.

available for runoff and percolation. Net precipitation is denoted as $P_{net,lct}(m)$. The four equations that complete the water balance are given by

$$E_{act,lct}(m) = \begin{cases} P_{lct}(m) - \left[K_{SM,lct}(m) - K_{SM,lct}(m-1)\right], & if\; P_{lct}(m) < E_{pot,lct}(m), \\ E_{pot,lct}(m), & if\; P_{lct}(m) \geq E_{pot,lct}(m), \end{cases} \tag{B-4}$$

$$P_{net,lct}(m) = max\left\{0,\; P_{lct}(m) - E_{act,lct}(m) - \left[K_{SM,lct}(m) - K_{SM,lct}(m-1)\right]\right\}, \tag{B-5}$$

$$DR(m) = c \cdot j \cdot \sum_{lct} A_{lct} \cdot P_{net,lct}(m), \tag{B-6}$$

$$P_{perc}(m) = c \cdot (1-j) \cdot \sum_{lct} A_{lct} \cdot P_{net,lct}(m). \tag{B-7}$$

where j is an exogenous fraction to the model, A_{lct} is the variable representing area of land (summed up over land cover types) and c is the factor for conversion from mm and area into billion tons. From here on all new variables are in billion tons.

Hydrological models typically feature nonconvexities in the process technology and this also holds for the AQUA model. In economics a convex *technology* is related to concave production *functions* and all other production functions, including convex functions, are said to correspond to nonconvexities in the technology. In case the production function is differentiable, a convex technology corresponds to a Hessian matrix of second-order partial derivatives that is negative semi definite. The functions in (B-2) and the first line of (B-3) fail this criterion, because their Hessians are positive semi definite. A third nonconvexity arises in equation (B-5) where the maximum of two linear functions is no longer a concave function. Finally, if land use is considered as being an endogenous variable, which is the case in the full AQUA model but not in our application, then the functions in (B-6) and (B-7) would also be strictly convex functions and induce another nonconvexity. In our application land utilization is a parameter and, therefore, these functions are linear and thus concave. Although the description of the AQUA model is not yet complete we already state that these are the only nonconvexities in AQUA.

The groundwater compartment

As Figure 1 illustrates, the groundwater compartment receives inflows from the soil moisture compartment in the form of percolation and return flows from the economy compartment, whereas outflow in the form of runoff goes to surface water compartment and human extraction to the economy compartment (there is no subsurface runoff from groundwater in the Upper-Zambezi). The monthly groundwater stock, denoted as $K_{GW}(m)$, states the quantity in store at the *end* of the month. The equations describing the groundwater compartment are entirely linear, because the runoff from groundwater is governed by the *storage principle*, meaning that the monthly runoff of groundwater to surface water, denoted as $R_{GW}(m)$, is a linear relation of last month's end stock of groundwater, denoted as $K_{GW}(m-1)$, and it is stated as

$$R_{GW}(m) = K_{GW}(m-1)/k_{GW} , \tag{B-8}$$

where the exogenous number k_{GW} can be thought of to represent the lag time of outflow. Human extraction of groundwater (per sector), denoted as $EH_{GW,sec}(m)$, and the return flow of human water use per sector, denoted as $RH_{GW,sec}(m)$, are specified in the economy compartment. Then the monthly change in the groundwater, denoted as $dK_{GW}(m)$, is simply given by

$$dK_{GW}(m) = P_{perc}(m) - R_{GW}(m) + \sum_{sec}\left(RH_{GW,sec}(m) - EH_{GW,sec}(m)\right). \tag{B-9}$$

and the evolution of the stock variable $K_{GW}(m)$ is equal to

$$K_{GW}(m) = K_{GW}(m-1) + dK_{GW}(m). \tag{B-10}$$

This completes the description of the groundwater compartment.

The surface water compartment

The third hydrological compartment of Figure 1 is the surface water compartment. This compartment receives inflows from the soil moisture compartment in the form of direct runoff, runoff from groundwater and return flows from the economy compartment, whereas outflow in the form of runoff goes to downstream (which leaves the Upper-Zambezi cell) and human extraction to the economy compartment. The monthly surface water stock, denoted as $K_{SW}(m)$, states the quantity in store at the *end* of the month. The equations describing this compartment are similar as the groundwater compartment, because its runoff to downstream is also governed by the storage principle. Using similar notation as for groundwater we have that runoff to downstream is given by

$$R_{SW}(m) = K_{SW}(m-1)/k_{SW} . \tag{B-11}$$

The monthly change in the surface water is the difference of in and outflows and it is equal to

$$dK_{SW}(m) = DR(m) + R_{GW}(m) - R_{SW}(m) + \sum_{sec}\left(RH_{SW,sec}(m) - EH_{SW,sec}(m)\right). \tag{B-12}$$

Finally, the evolution of the stock variable is given by

$$K_{SW}(m) = K_{SW}(m-1) + dK_{SW}(m). \tag{B-13}$$

This completes the description of the surface water compartment.

The economy compartment

To concentrate the analysis on the hydrological process, we drop the socio-economic relationships of the AQUA model and replace these by a list of end uses, such as monthly deliveries for irrigation at exogenously given prices, in order to describe a rudimentary model of the economy. The economy compartment receives its input from all three hydrological compartments, which is obvious for human extraction of ground and surface water. The economic contribution of soil moisture is motivated by agronomics, where plant growth is related to the availability of soil moisture. Incorporating the latter, although in a rather primitive way, is a novel feature in the valuation of hydrological processes, but its inclusion is necessary if all mechanisms that generate value are to be investigated. Without loss of generality, we keep the economy compartment linear.

As end uses of ground and surface water, AQUA considers irrigated agriculture, livestock, domestic use, industry and export out of the basin. Each of these five user categories (sectors) extracts water from surface water and groundwater. In our application we simply assume that

$$EH_{GW,sec}(m) = \overline{EH}_{GW,sec}(m) \text{ and } EH_{SW,sec}(m) = \overline{EH}_{SW,sec}(m). \quad \text{(B-14)}$$

Total extraction per sector is given by

$$EH_{sec}(m) = EH_{GW,sec}(m) + EH_{SW,sec}(m). \quad \text{(B-15)}$$

The return flows to the ground and surface water are fixed proportions of total extraction per sector that take into account water loss due to evaporation. Formally, the human induced return flows are given by

$$RH_{GW,sec}(m) = r_{GW,sec} EH_{sec}(m) \text{ and } RH_{SW,sec}(m) = r_{SW,sec} EH_{sec}(m), \quad \text{(B-16)}$$

where $r_{GW,sec}$ and $r_{SW,sec}$ denote exogenous sector-specific coefficients for return flows to the ground and surface water, respectively.

Three remarks are in order. First, it is easy to incorporate stock variables in the simple water related economy. Such stocks have to be present in semi-arid areas where water harvesting takes place and water is stored in reservoirs for use in dry periods. Second, the exogenous parameters specifying extraction are fixed at their steady state values, computed by simulating AQUA in the software package M, as described in Appendix E. This ensures that our steady state flows and stocks are similar to those in AQUA run under M. Third, in an extended model the right-hand side would be some kind of production function representing the 'pumping and transport' technology that requires other inputs such as energy and labor.

The final step in describing the economy compartment is to relate the water use to economic production. Within crop production we distinguish between rain-fed and irrigated agriculture, denoted by subscripts *Rf* and *Irr*. Plant growth requires soil moisture and irrigation adds additional soil moisture. Since incorporating the exact agronomic relation falls outside the scope of the present study, but can in principle be done, we propose the following linear relations for the economic value

$$V_{Rf} = \sum_m p_{SM,Rf} \cdot K_{SM,Rf}(m) \quad \text{(B-17a)}$$

and

$$V_{Irr} = \sum_m \left(p_{SM,Irr} \cdot K_{SM,Irr}(m) + p_{EH,Irr}(m) \cdot EH_{Irr}(m) \right), \quad \text{(B-17b)}$$

where V_{Rf} and V_{Irr} denote the monetary value of the marginal product (explained below) of rain-fed and irrigated agriculture, respectively. The summation for rain-fed agriculture (*Rf*) is over the length of growing period (January until May) and for irrigation (*Irr*) over the months in a two-cropping system irrigated crops grow (January till November). Following the AQUA model, irrigation water is equally distributed in the months April until November. For the other economic sectors the value generated is equal to

$$V_{sec} = \sum_m p_{EH,sec}(m) \cdot EH_{sec}(m). \quad \text{(B-17c)}$$

With respect to prices for end use in (B-17) we chose $p_{SM,Rf}(m) = P_{SM,Irr}(m) = 0.015$ per mm, $p_{EH,Irr}(m) = 1.5$ per billion ton and $p_{EH,sec}(m) = 2400$ per billion ton.

On the interpretation of the marginal product

The prices $p_{SM,Rf}(m), P_{SM,Irr}(m), P_{EH,sec}$ in equation (B-17) can be regarded as the sector specific willingness to pay for a slight increase of the deliveries by the hydrological system as input in the economic production. The reason is that in a more elaborated economy compartment a production function would be specified for each sector that links several inputs, including water, to economic output. The willingness to pay for an additional increase of water by each sector is equal to, what economists call, the marginal product of water in the production process and this marginal product measures the increase in the economic value of production if a small amount of extra water is applied to the current use of inputs. Formally, given output prices the marginal product of water consists of the price per unit of output times the partial derivative of the production function with respect to water. As an illustration, consider once more the numerical example of Appendix A that distinguishes upstream and downstream. In this example, we may reinterpret downstream as the economy compartment and then the marginal product of the economy is equal to $pg'(q)$ and, as seen before, this marginal product is equal to the shadow price f . In the objective function of the AQUA model we interpret the prices for deliveries to end users as sector specific shadow prices f . In a more elaborated economy compartment these (shadow) prices depend upon the level of inputs, and thus output, but in our application we simply chose fixed prices and this is without loss of generality. It also illustrates how value balances for hydrological processes and compartments are similar to those for standard economic production. To summarize, the prices $p_{SM,Rf}(m), P_{SM,Irr}(m), P_{EH,sec}$ should be regarded as the marginal product or shadow prices f associated with a more elaborated economy compartment.

Appendix C: The AQUA model formulated in the notation of Section 3

In this appendix the model of Appendix B is related to the notation of Section 3. This notation has the advantage that it identifies process-based hydrological models as a system of difference equations governing stocks responding to natural inflows. However, the notation might obscure the relation to the hydrological literature, where it is common to specify variables and functional forms in their recursive order. We start with identifying the vectors of natural resources, stock variables and human extractions and, then, the functions *H* and *F* in (3.1) and (3.2).

In the AQUA model hydrological cycles are calendar years represented by the index t with monthly time steps. Monthly precipitation and temperature per land cover type are the inflows of natural resources. Thus, the vector of natural recources g_t and vector of natural endowments w are seventy-two dimensional vectors and correspond to equation (B-1).

The hydrological system describes the evolution of the hydrological stock variables over an annual cycle given the inflow of natural resources and human induced flows. The three hydrological compartments distinguish five stock variables per month, namely for each land cover type the soil moisture $K_{SM,lct}(m)$, the groundwater stock $K_{GW}(m)$ and the surface water stock $K_{SW}(m)$. Therefore, the vector of stocks per annual cycle is sixty dimensional and the associated index *j* represents sixty different locations.

The vector d_t in section 3 consists of all human induced extractions and return flows, differentiated with respect to month, sector and water source. These are $EH_{GW,sec}(m)$, $EH_{GW,sec}(m)$, $RH_{SW,sec}(m)$ and $RH_{SW,sec}(m)$, which are two hundred and forty variables. Since crop production is also included as an economic activity the extraction of moisture from the soil should also be included in the vector d_t. The price for return folws is zero in the objective.

Hydrological systems have stocks and flows. At first glance, the abstract representation of (3.1) seems to ignore flows, except for inflows of natural resources and human induced flows. However, (3.1) should be seen as the reduced form of a process-based model that can in principle be obtained through recursive substitution. Beside the impossibility of such an exercise, we certainly do not want to advocate such elimination, because it would obscure flows that represent deliveries between compartments needed to identify where accounts might be separated. In AQUA the equations (B-2)-(B-13) specify the evolution of stocks and flows and, therefore, these equations implicitly specify the ‘reduced form’ function *H*.

Similar as for the function *H*, the function *F* is the reduced form of all variables and equations that determine human induced flows and crop growth. In AQUA these variables and equations consist of (B-14)-(B-16). Note that the function *F* in our application does not depend upon the stocks k_t or the natural resources g_t.

One final remark, although all variables in AQUA change over time, the functional forms do not. Therefore, these functional forms represent an autonomous system of difference equations.

Appendix D: Implementing accounting in AQUA

In this appendix the AQUA model as described in Appendix B is modified for accounting purposes. The modifications are fourfold. First, the AQUA model should be brought under an optimization, which requires adding an overall objective function. Second, structural factors should be added to the AQUA model in order to value the contribution of process technologies. Third, the separation of compartments or zooming in on processes requires the introduction of accounting variables. Fourth, the numerical procedure for valuation in the steady state has to be implemented. Each of these steps is discussed in this Appendix.

The objective function

The objective function is kept simple and consists of the maximization of the economic value of production, although the amenity value of ecological variables could also be included as long as these can be expressed through a functional form. In our application, value is generated by crop production, other economic activities and by downstream users. Downstream hydrological processes are not yet explicitly modeled and, therefore, we assume that downstream users simply receive the runoff from surface water $R_{SW}(m)$ and generate value out of it. The annual value of downstream, denoted as V_{DS}, is equal to

$$V_{DS} = \sum_m p_R(m) \cdot R_{SW}(m), \qquad \text{(D-1)}$$

where $p_R(m)$ is the monthly marginal contribution of monthly runoff. We took $p_R(m) = 0.1$ per billion ton during the wet season and $p_R(m) = 1$ per billion ton during the dry season in order to reflect that river water is most productive in dry season. Note that in a multi-cell application downstream value always enters upstream cells through runoff to downstream (and subsurface runoff if present). The objective function is given by the sum of (B-17) and (D-1), i.e.

$$V_{DS} + V_{Rf} + V_{Irr} + \sum_{sec} V_{sec}. \qquad \text{(D-2)}$$

We note that in order to translate the objective function into the terminology $p'd_t$ the variables $R_{SW}(m)$ should be added to the vector representing human activities.

Implementing structural factors

According to the discussion in Appendix C we have that the equations (B-2)-(B-13) represent the function *H* and (B-14)-(B-17) for the function *F*. Functional forms that are not homogeneous of degree one require a structural factor. For completeness, the function $g:\mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree one if $g(\lambda x) = \lambda g(x)$ for every positive λ. Application of this definition to the linear function $g(x) = a'x + b$ yields that $g(\lambda x) = \lambda a'x + b \neq \lambda g(x)$ unless the constant term b is zero; an insight that will return below since most equations of the AQUA model are linear. Adding the scalar z as the structural factor means that the redefined function $\tilde{g}$ becomes $\tilde{g}(x,z) = zg(x_1/z,\dots,x_n/z) = a'x + bz$ and this function is homogeneous of degree one in all of its variables because $\tilde{g}(l\,x, l\,z) = l\,\tilde{g}(x,z)$. It is important to note that there are two simple cases for linear functions that do not require adding a structural factor. First, in case the constant term b is zero, because then the structural factor cancels out. Second, in case of the constant function $g(x) = b$ it can be mathematically shown that the shadow price associated with $z = 1$ will be equal to b times the shadow price of the equation $g(x) = b$.

The *soil moisture* compartment is the natural starting point for starting the implementation of structural factors and we maintain the same order of equations as before. The functional form on the right-hand side of equation (B-2) is a strictly convex function that is not homogeneous of degree one. In the application month and land cover type specific structural factors are implemented, denoted by $z_{PE,lct}(m)$. Then (B-2) is replaced by

$$E_{Pot,lct}(m) = c_{lct}(m) \cdot T_{lct}(m)^{a} \cdot z_{PE,lct}(m)^{1-a} \qquad \text{(D-3)}$$

and $z_{PE,lct}(m) = 1$ is added to the model equations, which extends the AQUA model by thirty-six variables and equations. Note that there is some degree of freedom for the modeler. The advantage of monthly and cover type specific structural factors is that value accounts can be separated to the most disaggregated level possible: the level of a single equation. For (B-2) this means that we can specify a value account for each month and cover type where the 'delivery' potential evaporation is related to its 'source' temperature and the contribution of the conversion technology. However, in situations where the size of the model is an issue, the number of structural factors (and equations) can be limited to three annual structural factors per cover type or to a single factor, but then the annual value account per cover type for (B-2) cannot be further separated on a monthly basis. Moreover, a single structural factor for all months and all cover types would not allow separating the cover types and months. It is up to the modeler and the policy maker which level of separation is most wanted. Similar considerations apply to all further equations, although for some model equations the choice for constructing value accounts at the equation level would also require the adding of accounting variables and equations, a topic that is addressed separately.

The second functional form that requires a structural factor for its three variables $E_{Pot,lct}(m), K_{SM,lct}(m-1), P_{lct}(m)$ is equation (B-3). To see this, note that exponential function is not homogeneous of degree one in case potential evaporation exceeds precipitation and, for the other case, the constant $\overline{K}_{KM,lct}$ under the minimum corresponds to the constant linear function without a slope. The redefined function with the structural factor as $z_{K,SM,lct}(m)$ becomes

$$\begin{cases} K_{SM,lct}(m-1)\; e^{\left(P_{lct}(m) - E_{Pot,lct}(m)\right) / \left(\overline{K}_{SM,lct} \cdot z_{K,SM,lct}(m)\right)}, \\ min\left\{K_{SM,lct}(m-1) + P_{lct}(m) - E_{Pot,lct}(m), \overline{K}_{SM,lct} \cdot z_{K,SM,lct}(m)\right\}, \end{cases} \qquad \text{(D-4)}$$

and $z_{K,SM,lct}(m) = 1$. Similar as for (D-3), the AQUA model is extended with another thirty-six variables and equations.

The functions in (B-4), (B-6) and (B-7) are linear in the variables without a constant term and, therefore, these functions are already homogeneous of degree one. Although the maximum function in (B-5) is not everywhere homogenous of degree one it describes a piecewise linear function. Since both linear parts do not have a constant term the structural factor cancels in both parts and can therefore be neglected. This completes adding structural factors to the soil moisture compartment and, to summarize, only equations (B-2) and (B-3) needed some modification to include these factors.

The *groundwater* compartment consists of entirely linear equations without a constant term. Therefore, all equations in this compartment are already homogeneous of degree one and there is no need to add structural factors. This means that this compartment only passes through value, either directly with its outflows or stored in the groundwater reservoir. The hydrological process seen as a production technology does not contribute in generating value. Note that this insight holds for any compartment governed by the storage principle. The *surface water* compartment is similar to the groundwater compartment and also governed by the storage principle. Therefore, the same conclusions hold as in the groundwater compartment.

The final compartment is the *economy* compartment and it also consists of linear equations that do not require structural factors. Note that there are two cases. First, equations (B-15)-(B-17) are linear without a constant term. Second, (B-14) consists of the constant function and, thus, the shadow price of the structural factor is equal to the shadow price of the constant term times the shadow price of (B-14). The contribution of the structural factor should be thought of as the marginal contribution of other economic production factors such as the pumping technology and energy costs. In terms of the notation of section 3 and linearity, we can specify a vector of constants $\bar{d}$ representing all the constants in equation (B-14)-(B-17), including all zeros, and then the aggregate value associated with structural factors would be $q'\bar{d}$.

Implementing accounting equations: endowments and stocks

The main purpose of accounting variables and equations is to generate correct shadow prices for stocks and flows. The accounting equation can be regarded as a measuring device for value, where this value consists of the shadow price times the quantity represented by the accounting variable. As mentioned in section 3, accounting equations for valuing endowments and stocks are always needed and these are associated with $g_t = w$ and $\tilde{k}_t = k_t$, with l_t, respectively, y_t as the associated vector of shadow prices.

In Appendix C monthly precipitation and monthly temperature are the only natural endowments identified in the AQUA model and (B-1) is already of the form $g_t = w$. From a conceptual point of view we might interpret constant terms appearing in the functional forms as representing natural endowments. For example, the soil moisture carrying capacity $\bar{K}_{SM,lct}$ in (D-4) regarded as a natural endowment represents the physical characteristics of the soil structure and its vegetation. As an alternative to measure its value through the structural factor, we can introduce the accounting variable $\hat{K}_{SM,lct}(m)$ and accounting equation $\hat{K}_{SM,lct}(m)=\bar{K}_{SM,lct}$ to measure this endowment. In the application there was no advantage in pursuing this line, but with multiple constants in one functional form one might be interested in separating the effects of several constants.

The implementation of the accounting equation $\tilde{k}_t = k_t$ in AQUA can be done without introducing additional variables $\tilde{K}_{SM,lct}(m)$, which follows from two observations. First, the evolution of the hydrological processes in AQUA is on a monthly basis and to let the system evolve for another month it suffices to know only last month's stocks.[14] Since the valuation method applied to AQUA is over annual cycles we only need to incorporate the stocks of last year's December. Second, the infinite time horizon problem (3.4) is broken down into an infinite series of annual valuation problems (3.5) and the quantities of stock carried over from one cycle to the other are known through simulation of the model and, thus, can be treated as constants in problem (3.5). So, instead of introducing new variables $\tilde{K}_{SM,lct}(m)$ for all months and going through the trouble of replacing all existing variables $K_{SM,lct}(m)$ by accounting variables, we maintain the last variables and introduce constants that represent the past stocks. In terms of the notation of Section 3, we write k_t for $\tilde{k}_t$, introduce the constant $\bar{k}_t$ representing the last cycle's stocks and add the equation $k_t = \bar{k}_t$ to problem (3.5). Combining the two remarks while denoting December's stocks as $m=0$ means that we introduce the constants $\bar{\bar{K}}_{SM,lct}, \bar{K}_{GW}, \bar{K}_{SW}$ and the accounting equations

$$K_{SM,lct}(0) = \bar{\bar{K}}_{SM,lct}, \quad K_{GW}(0) = \bar{K}_{GW} \quad and \quad K_{SW}(0) = \bar{K}_{SW}, \tag{D-5}$$

with the remark that the constants are last year December's stocks obtained through simulation. Implementation of problem (4.1) requires further modification of (D-5), which is addressed separately. The shadow prices of (D-5) yield the valuation of the initial stocks at the beginning of the cycle according to problem (3.5), provided the shadow prices y_{t+1} are known.

Implementing accounting variables: separation of processes and compartments

The separation of processes and compartments also requires accounting variables and equations. In Section 5 value balances are constructed for each of the three hydrological compartments and the economy compartment. Before paying attention to these compartments we first discuss a general principle when accounting variables are not necessary to separate accounts and when they are necessary.

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[14] In the terminology of system or control theory, the state of the system coincides with last month's stock variables. In general, it always suffices to include only the state variables in terms of system theory in the implementation of accounting equations for the valuation of stocks.

The recursive structure of process-based models implies that exactly one model equation can be identified to each model variable and in this equation the model variable appears on the left-hand side. The rationale is that in running a simulation each variable obtains a value in quantity terms and this occurs only once in the simulation. Furthermore, each model variable may appear more than once on the right-hand side of several model equations. For example, runoff from groundwater in (B-8) affects the groundwater reservoir in (B-9) and the inflow of surface water in (B-12). The general principle is that in case a variable appears only once on the right-hand side of all model equations then this variable's equation, where the variable appears on the left-hand side, can already be regarded as an accounting equation, because the effect of passing through the physical quantity affects the rest of the system in an unambiguous manner and the shadow price of this variable's equation's reflects the correct value per unit. However, in case a variable appears on the right-hand side of more than one equation, then the shadow price of this variable's equation measures the combined marginal effect of this variable upon the rest of the system through different channels and to single out one or more of these effects in order to separate accounts requires introducing accounting variables and equations. So, in our example of groundwater runoff the delivery from the groundwater compartment to the surface water compartment requires an accounting variable and equation. In discussing accounting issues we first focus upon the relatively easy groundwater compartment, then on the surface water compartment and, finally, upon the soil moisture compartment. Then the economy compartment is also automatically separated.

The *groundwater* compartment receives percolation from the soil moisture compartment, delivers groundwater to the surface water compartment and delivers possibly negative quantities of net extraction to the economy compartment. With respect to percolation, this variable appears exactly once on each side of the model equations, namely in (B-7) and (B-9). So, the shadow price of (B-7) correctly values the monthly delivery between these two compartments. Groundwater runoff $R_{GW}(m)$ appears twice on the right-hand side in (B-9) and (B-12). In this case, the shadow price of (B-8) measures two opposite effects on welfare, namely the value per billion ton delivered to surface water and the (negative) effect of extracted value from the groundwater reservoir. In order to correctly value the delivery to surface water we introduce the accounting variable $\hat{R}_{GW}(m)$, the accounting equation

$$\hat{R}_{GW}(m) = R_{GW}(m) \qquad \text{(D-6)}$$

and substitute $\hat{R}_{GW}(m)$ into (B-12) to obtain

$$dK_{SW}(m) = DR(m) + \hat{R}_{GW}(m) - R_{SW}(m) + \sum_{sec}\left(RH_{SW,sec}(m) - EH_{SW,sec}(m)\right). \qquad \text{(D-7)}$$

Accounting equation (D-6) correctly measures the shadow price of monthly deliveries made to the surface water compartment and this shadow price is used to construct Table 4 in Section 5.

Note that the negative effect per billion ton on the value stored in the groundwater reservoir is the shadow price of (B-8) minus the shadow price of (D-6). Figure D-1 illustrates these two effects in our application (the monthly value extracted seems constant but actually fluctuates slightly). Finally, additional accounting variables and equations are also needed to correctly value the deliveries of human extraction $EH_{GW,sec}(m)$, because these variables appear on the right-hand side of (B-9) and (B-15). In the application this is implemented by incorporating aggregate extraction from groundwater, denoted as $\hat{E}_{GW}(m)$, adding the accounting equation $\hat{E}_{GW}(m)=\sum EH_{GW,sec}(m)$ and substitution of $\hat{E}_{GW,sec}(m)$ for the sum of extractions in (B-9).

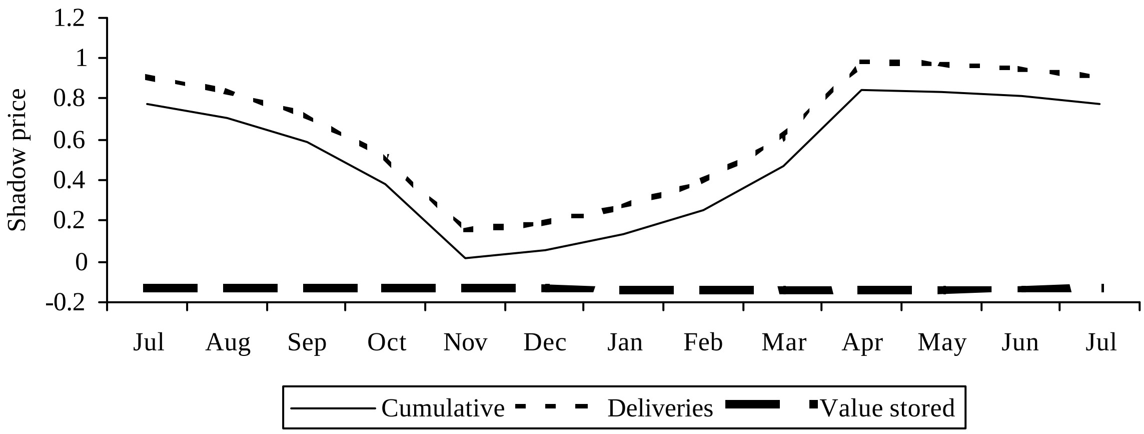

Figure D-1 The shadow price of (B-8) is the cumulative of the shadow price for deliveries to surface water in (D-6) and the value extracted from the reservoir. All prices are per billion ton.

The separation of the value account for the *surface water* compartment requires two adjustments similar as for the groundwater compartment. With respect to surface water runoff that is delivered to downstream users in (D-1): Introduce the accounting variable $\hat{R}_{SW}(m)$, the accounting equation $\hat{R}_{SW}(m)=R_{SW}(m)$ and substitute $\hat{R}_{SW}(m)$ for $R_{SW}(m)$ in equation (D-1). Similar, adding aggregate extraction from surface water is necessary to separate the surface water compartment from the economy compartment.

The water balance of the *soil moisture* compartment water balance has deliveries in the form of evapotranspiration, direct runoff, percolation and end stocks, and as its sources natural endowments, structural factors and initial stock. With respect to the deliveries of direct runoff and percolation to the ground and surface water compartment no accounting variables and equations are necessary. However, the deliveries to the economy compartment through the stock variables in (B-17) require the accounting variables $\hat{K}_{SM,Rf}(m)$ and $\hat{K}_{SM,Irr}(m)$ for the appropriate crop growing months, the accounting equations

$$\hat{K}_{SM,Rf}(m)=K_{SM,Rf}(m) \quad \textit{and} \quad \hat{K}_{SM,Irr}(m)=K_{SM,Irr}(m) \tag{D-8}$$

and substitution of the accounting variables in (B-17).

Finally, as mentioned, having separated each of the three hydrological compartments implies that the economy compartment is also separated from these modules.

Implementation in the steady state

The adjustments discussed thus far implement problem (3.5) and it requires one more step to implement the steady state procedure of problem (4.1). In the notation of Section 3 and 4, these steps require replacing k_t by $rk_{t+1} + (1-r)k_t$, adjusting the accounting equations for valuing stocks and correct initialization of steady state stocks. The last two steps boil down to adjusting (D-5) into

$$(1-r)K_{SM,lct}(0) = \bar{\bar{K}}_{SM,lct}, (1-r)K_{GW}(0) = \bar{K}_{GW} \text{ and } (1-r)K_{SW}(0) = \bar{K}_{SW}, \qquad \text{(D-9)}$$

and initializing the constants on the right-hand side as follows

$$\bar{\bar{K}}_{SM,lct} = (1-r)K^*_{SM,lct}, \quad \bar{K}_{GW} = (1-r)K^*_{GW} \quad \text{and} \quad \bar{K}_{SW} = (1-r)K^*_{SW}, \qquad \text{(D-10)}$$

where the star denotes the steady state values in December.

The implementation of the first step requires that last year's December stocks are replaced by a weighted combination of these stocks and this year's December stocks. Formally, this means that we have to replace in every model equation $K_{SM,lct}(0)$ by $rK_{SM,lct}(12) + (1-r)K_{SM,lct}(0)$ and similar for $K_{GW}(0)$ and $K_{SW}(0)$. This substitution was implemented in equations (D-4), (B-4), (B-5), (B-8), (B-10), (B-11) and (B-13) for the month January, i.e. $m=1$.

Appendix E: The software migration of AQUA

The AQUA model is embedded, calibrated and simulated in the software package M. The language M is developed at the (Dutch) National Institute of Public Health and the Environment[15] as an interface to run numerical simulations of large-scale process-based models based upon differential equations. In M the modeler can set the numerical method to approximate the variables' time paths, including the Euler or first Runga-Kutta (RK-1) method and the second Runga-Kutta (RK-2) method. We refer to Appendix F for a discussion of these methods. In Hoekstra (1998) the RK-2 method with a two-week time step is applied for both calibration and simulation. The model parameters in the application reported in Section 5 are based upon this calibration. The RK-2 interpolates once, which makes it effectively a weekly basis, but the M-software does not give these intermediate results.[16] Also it is not clear how M translates monthly data into weekly data if run on a two-week basis with one interpolation step.

The computation of shadow prices is implemented in the optimization software GAMS, which is flexible enough to run simulations also. In translating AQUA into GAMS the RK-1 method on a monthly time basis is implemented. Calculus transparency and tractability are the reasons for this. As discussed in Appendix F, this simplification is not important in order to demonstrate the validity of the value balance approach, even though the RK-2 method could also be implemented in GAMS, provided the software package M would allow for obtaining results of intermediate steps to test correct implementation in GAMS. The three hydrological compartments were transferred to and translated into GAMS. A simulation performed in GAMS reproduced the same steady state as in AQUA run under M with the RK-1 method turned on. The numerical precision of the intermediate data file produced by M is unfortunately low, which complicated the interpretation of observed small numerical differences in steady state values between M and GAMS.

A theoretical exercise performed on the set of equations governing the groundwater storage and its system of water balances, where monthly inflow is fixed at its steady state values, revealed a unit-root problem known from time series analysis, e.g., see Green (1993). To see this, consider the simple (one-dimensional) difference equation $v_{t+1} = av_t + b$, which is stable for $-1 < a < 1$. Linearity and stability imply convergence to the unique steady state $v^* = b/1 - a$. A unit-root problem corresponds to $a \approx 1$ and, then, $v_{t+1} - v_t = -(1-a)v_t + b$ for $1 - a \approx 0$ dampens the contribution of v_t, which implies that the convergence to the steady state becomes slow. In AQUA the outflow of groundwater is governed by the storage principle, i.e., monthly runoff from groundwater is equal to ck_m, where k_m denotes the stock of groundwater in the *m*-th month and c a positive parameter. So, on a monthly basis, we have that $k_m = (1-c)k_{m-1} + in_m - out_m$. Since the calibrated c is 0.0012 a unit-root problem arises across years. The associated slow convergence is confirmed by the approximately 1500 annual cycles in GAMS needed to reach the steady state. Implementing an analytical formula for the steady state value of the groundwater

[15] For more information we refer to www.rivm.nl.

[16] In the terminology of Appendix F, M does not give the values of the variable *y* in (F-2).

reservoir, which is similar to the one for $v*$, circumvented the long simulation time. The equivalent of the constant b in the formula consists of inflow from the soil moisture compartment, which reaches its steady state within a couple of years, and the (by our simplifying assumption) constant in- and outflow of the economy compartment. The calculated steady state for groundwater slightly deviates from the M results.[17] Numerical differences with M are probably caused by a combination of the imprecision in the intermediate data-files from M and not having reached the steady state in M after sixty years. Finally, the surface water is also governed by the storage principle and the same arguments apply to the set of difference equations governing the surface water, but the unit-root problem does not show up. Nevertheless, we implemented a similar formula as for groundwater, in order to speed up the simulation to obtain the steady state values.

[17] The steady state simulated in M under RK-1 reveals a minor difference between the current stock in December and next year's stock in December in the third digit. In GAMS we iterated until differences between this and last year's reference month December vanished.

Appendix F: Discrete time approximation of differential equations

Application of numerical software packages to simulate the time path of a system of differential equations is based upon some discrete time approximation of the unknown time path. Two popular approximations implemented in many packages, including the package M, are the Euler or first Runga-Kutta (RK-1) method and the second Runga-Kutta (RK-2) method. These two methods are discussed in this appendix in order to show that our choice for the simpler RK-1 method instead of the RK-2 method in the model description of AQUA in Appendix B is not important to demonstrate the validity of the proposed valuation method.

Consider an arbitrary system of differential equations given by $\dot{x}(t) = f(x(t))$ and the initial condition $x(0) = x_0$, where the function f is continuous. Denote the size of each time step in the discrete time approximation as $D > 0$ and consider the (m+1)-th time step, that is consider time $t = (m+1)D$ in the differential equation. The RK-1 method approximates the time path as

$$x((m+1)D) = x(mD) + D \cdot f(x(mD)). \qquad \text{(F-1)}$$

The (m+1)-th time step of the RK-2 method requires the introduction of the auxiliary variable y(m+1) and an auxiliary equation such that

$$\begin{aligned} y(m+1) &= x(mD) + 0.5D \cdot f(x(mD)), \\ x((m+1)D) &= x(mD) + D \cdot f(y(m+1)), \end{aligned} \qquad \text{(F-2)}$$

which can in principle be reduced to

$$x((m+1)D) = x(mD) + D \cdot f(x(mD) + 0.5D \cdot f(x(mD))). \qquad \text{(F-3)}$$

In implementing the RK-2 method the modeler has some degree of freedom and it depends upon the size of the model and the complexity of the function f whether to opt for (F-2) or (F-3). Nevertheless, both the RK-1 and RK-2 method are easy to implement.

With respect to value *accounting* it is important to note that there is a potential pitfall when adding structural factors to (F-2), because the auxiliary variable and its equation have no physical interpretation and are only mathematical constructs of the approximation method. In order to capture the technological contribution underlying the differential equation, both equations in (F-2) should get the *same* structural factor, which is in principle similar to assigning a structural factor to the reduced form (F-3).

Appendix G: Non-differentiability and shadow prices

The valuation method assumes the partial differentiability of the functions H and F, but not all hydrological processes can be described by such functions. In this appendix the problem of non-differentiable functions is addressed and it is argued that from a theoretical perspective such functions may give rise to a multiplicity of shadow prices, but that this problem is unlikely to exist in case of standard hydrological functional forms. However, there is some concern in case of economic behavior and the occurrence of complementarities in production .

Before discussing why non-differentiable functions might pose a problem, we first consider the graph $(f(x),x)$ of a partially differentiable function f of x. In each point of the graph of a differentiable function there exists a unique hyperplane that is tangent to the graph and perpendicular to the gradient of the function. Furthermore, each partial derivative uniquely represents the marginal contribution of the associated variable. For a function that is almost everywhere differentiable (but not everywhere) nondifferentiable points coincide with points of the graph where multiple hyperplanes are tangent to the graph and each hyperplane is perpendicular to a vector called the *subdifferential.* As before, each subdifferential represents a vector of marginal contributions, but the selection by the modeler of one of these subdifferentials as 'the' vector of marginal contributions has become an arbitrary choice. The multiplicity of subdifferentials carries over to the shadow prices through the first-order conditions. To see this, consider the maximization problem given by

$$\begin{aligned} &\max f(x) \\ &\text{subject to} \\ &g(x)=b, \quad (l) \end{aligned}$$

where f and g are differentiable almost everywhere. Denote subdifferentials by ∂, then the first-order conditions expressed as subdifferentials become

$$\begin{aligned} \partial f(x) - l\partial g(x) &= 0, \\ g(x) &= 0. \end{aligned}$$

Obviously, if both functions are partially differentiable in the optimum, then the shadow price λ will be unique. However, in case one of the functions is not differentiable in the optimum, there will be one shadow price λ for every combination of subdifferentials for f and g. Thus, the multiplicity in subdifferentials in the optimum carries over to the shadow prices and the contribution per unit to the objective has become ambiguous.

The multiplicity problem just sketched is hypothetical in hydrological applications, because most functional forms in hydrology are differentiable and, otherwise, differentiable almost everywhere. In process-based models non-differentiability typically arises in one of the following two cases: Whenever the maximum or minimum over two (or more) differentiable functions is taken or whenever the mathematical description of the process distinguishes different regimes.

For example, in the AQUA model (B-5) corresponds to the first category, (B-4) to the second and (B-3) simultaneously belongs to both categories. Note that the nondifferentiability in (B-5) can only arise in the case that the second term under the maximum function equals zero, which requires for instance the improbable amount of precipitation that is exactly equal to the sum of the other variables. Similar, the nondifferentiability in (B-4) only occurs at the regime switch and also requires the improbable circumstance that the levels of precipitation and potential evaporation coincide. For the nondifferentiabilities in (B-3) a combination of these arguments applies. The insight that nondifferentiability is improbable generalizes to process-based models in general, especially, if it is also taken into account that the functional forms depend upon calibrated coefficients and these should also be of the 'correct' value in order to result in non differentiable functions.

In the application a check was implemented to detect these improbable events, but as expected these did not occur. Moreover, before the valuation was run, the simulation run was used to automatically select the prevailing maximum or regime in (B-3)-(B-5) through the use of 0/1 variables. So, the valuation is based upon the prevailing functional forms and these are differentiable. This procedure avoided application of dedicated solvers in GAMS that can effectively handle 'nondifferentiable' problems.

There are however some concerns for nondifferentiable functions in especially the economy compartment in case some of the production processes involve complementarities, meaning the impossibility of substituting inputs to maintain a certain level of production. For example, in the production of bottled beverages, each one-liter bottle requires one liter of liquid to fill it and more bottles cannot substitute for the beverage and vice versa. So, the number of (filled) bottles is the minimum of the number of available bottles and the amount of beverage (in liters), i.e. $min\{x_1,x_2\}$. The management of the beverage firm maximizes profits by not spilling liquid when filling bottles, meaning that it aims at an input level of one liter liquid per bottle. Hence, economic behavior with complementarities in production drives the use of economic inputs to the point where nondifferentiability is the issue. This problem especially can arise in modeling agricultural production. Assuming e.g. a von Liebig crop growth the first limiting input (moisture and nutrients) determines the output. Then similar as for the beverage industry, economic application of irrigation water and chemicals leads to input levels where the production function is not differentiable. It is for these types of production processes inside firms that accountancy fails the answer of how to value the contribution of separate inputs and any division of costs will be based upon an arbitrary choice . Our method is no exception to this rule.

At a higher level of abstraction, this discussion shows that the mathematical demands upon the functional forms to obtain the values for the (physical) primal variables through simulation of the model are less demanding than the conditions needed to arrive at unique shadow prices.[18] Since complementarities in hydrological and economic production processes cannot be ruled out there always remains some concern for multiplicity of shadow prices. Optimization software such as GAMS are designed to calculate one of possible many shadow prices in such case and, if the modeler is not aware of the this computational issue, he or she may not even notice that the valuation is based upon an arbitrary choice made by the software. A good understanding of the mathematical problem and the numerical routines run by the software are necessary for a proper interpretation of shadow prices calculated.

[18] This is due to the nonlinearity of the functional forms. To see this, first consider the linear case $H(x) = Ax = H'(x)x$ for some triangular matrix A. Then the water flow follows gravity and value moves against gravity. A regular matrix A and no flows to 'absorbing states' such as unutilized fossil aquifers.is a sufficient condition for unique shadow prices. For nonlinear processes, $H(x) \neq H'(x)x$ and the conditions on F for continuous flows and no permanent absorption differ from those for unique shadow prices. In case of differentiable functions, a regular matrix of nonzero partial derivatives associated with positive water flows suffices. For further details, we refer to Keyzer (2002).

The Centre for World Food Studies (Dutch acronym SOW-VU) is a research institute related to the Department of Economics and Econometrics of the Vrije Universiteit Amsterdam. It was established in 1977 and engages in quantitative analyses to support national and international policy formulation in the areas of food, agriculture and development cooperation.

SOW-VU's research is directed towards the theoretical and empirical assessment of the mechanisms which determine food production, food consumption and nutritional status. Its main activities concern the design and application of regional and national models which put special emphasis on the food and agricultural sector. An analysis of the behaviour and options of socio-economic groups, including their response to price and investment policies and to externally induced changes, can contribute to the evaluation of alternative development strategies.

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