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Solving an intertemporal Arrow-Debreu model under aggregate risk: implementation of an SQG-based algorithm

by

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Abstract

The paper describes the implementation of a solution algorithm for a two-period, multi-commodity, multi-actor general equilibrium model with aggregate risk. Uncertainty is reflected by commodity-specific shocks, affecting the actors differently and leading to stochastic period-2 prices. The model has contingent markets of the Arrow-Debreu type, with constraints on borrowing and insurance. The algorithm is based on a proposal by Ermoliev and Keyzer to rewrite the model in Negishi-format and apply techniques from stochastic optimization. It has a deterministic inner loop for period 2, a stochastic-optimization middle loop for period 1 and a Negishi outer loop for intertemporal budgets. The current implementation, using standard functional forms for utility and investment, functions well. In the end, it appears to be mainly the speed of the inner loop that determines the success of the algorithm. Extension to more periods seems possible, via recursive methods, if the multiple inner loops can be solved rapidly.

Section 1
Introduction[1]

Market clearing in an intertemporal applied general equilibrium (AGE) model with aggregate risk leads to stochastic future prices. Since economic uncertainty comes from a large or even infinite number of events and standard methods of solving an AGE model require price iterations for each future event, application of a standard solution method would be cumbersome for reasons of dimensionality. Therefore, Ermoliev and Keyzer (1998) propose to rewrite such a model in Negishi-format which can be solved via iteration over (non-stochastic) welfare weights instead of prices. Furthermore, they advocate the application of stochastic quasi-gradient (SQG) techniques to compute optimal allocations under uncertainty, as opposed to traditional Monte-Carlo simulation.

In this paper we describe the implementation of the Ermoliev-Keyzer proposal for a two-period closed AGE model with an infinite number of possible future events (drawn from continuous density functions), contingent Arrow-Debreu markets and liquidity constraints. The latter are divided into borrowing constraints and insurance constraints. The model is applied in a setting with several actors and commodities. A random event consists of the outcomes of a group of commodity-specific endowment shocks in period 2. The current paper explains the algorithm and its functioning. Van Veen (2001) describes the economic context of the simulations, focusing on saving and investment decisions.

The solution of the model can be characterized by a set of necessary and sufficient conditions which form the basis for the algorithm. The algorithm consists of three loops: an inner loop for the period-2 allocation, a middle loop for the period-1 allocation and an outer loop for the intertemporal budgets. Thanks to the SQG-approach the inner loop need not consider more than one random event at each update of the period-1 allocation. The budget balances are obtained via adjustment of the welfare weights. Theoretically, existence of the equilibrium can be proved only if the number of events is finite or the insurance constraints are not binding. With binding insurance constraints and an infinite number of events, existence of the equilibrium is not guaranteed. In this case, the algorithm must serve also as (pragmatic) existence proof.

Section 2 presents the model in the traditional format, the excess-demand format. In section 3 the alternative Negishi formulation is given. Section 4 discusses the equivalence between the two formats, and derives the conditions on which the algorithm is based. After some general algorithmic background in section 5, the algorithm is spelled out in section 6. The functional forms of the actual application are specified in section 7. Section 8 describes the experiences with respect to convergence and solution time. Finally, section 9 evaluates the method and addresses the extension to more periods.

[1] The author thanks Michiel Keyzer for his advice in implementing the algorithm, and Chris Elbers and Jan-Willem Gunning for their comments on an earlier version of the paper.

Section 2
The model

Here, we present the model in excess-demand format, discuss existence of the solution, and list the first-order conditions. The formulation is the same as in Van Veen (2001). Annex A contains the full list of symbols.

There are m actors, indexed i, and r commodities, indexed k. The time subscript is t. In period t actor i's consumption is represented by r-vector x_{it}, the endowments by r-vector ω_{it}, the investment level by scalar d_{it}, net transfer receipts by scalar $\bar{t}_{it}$, income by scalar h_{it} and the budget deficit by scalar V_{it}. The r-vector p_t stands for the prices in period t. The symbols u_{it} denote the instantaneous utility functions. Uncertainty in period 2 is reflected by the random event ε_2 which is an N-vector of random sources, indexed n, with joint density function f. Household behaviour is represented as maximization of expected utility subject to the budget constraint and liquidity constraints. The actors agree about the commonly believed density function f but in the formulation of expected utilities they may follow a more subjective attitude, expressed via own density functions f_i. In equations, with shadow prices of the constraints between brackets:

$$\max_{x_{i1}, x_{i2}(\varepsilon_2), d_{i1} \geq 0} \quad u_{i1}(x_{i1}) + \int u_{i2}(x_{i2}(\varepsilon_2)) f_i(\varepsilon_2 \mid I_1) d\varepsilon_2 \tag{2.1}$$

$$\text{s.t.} \quad V_{i1} + \int V_{i2}(\varepsilon_2) f(\varepsilon_2) d\varepsilon_2 \leq 0 \qquad (\lambda_i)$$

$$V_{i1} \leq Z_{i1} \qquad (\mu_{i1})$$

$$V_{i2}(\varepsilon_2) \leq Z_{i2} - V_{i1} \qquad (\mu_{i2}(\varepsilon_2))$$

in which $h_{i1} = p_1(\overline{\omega}_{i1} + \bar{t}_{i1}\vartheta)$

$$V_{i1} = p_1(x_{i1} + \eta_i d_{i1}) - h_{i1}$$

$$Z_{i1} = \ell_{i1} h_{i1}$$

$$Z_{i2} = \ell_{i2} h_{i1}$$

$$\omega_{i2} = \overline{\omega}_{i2}(\varepsilon_2) + g_i(d_{i1}, \varepsilon_2)$$

$$h_{i2}(\varepsilon_2) = p_2(\varepsilon_2)(\omega_{i2} + \bar{t}_{i2}\vartheta)$$

$$V_{i2}(\varepsilon_2) = p_2(\varepsilon_2)(x_{i2}(\varepsilon_2) + \eta_i \overline{d}_{i2}) - h_{i2}(\varepsilon_2)$$

Household decisions and prices in period 2 are not deterministic but depend on the random event, since the realizations are assumed to be known in period 2. The dependence is made explicit by writing the variables as functions of the random event. The scalars Z_{i1} and Z_{i2} denote the upper bound on actor i's borrowing and insurance, respectively. These bounds are imposed via exogenous fractions ℓ_{it} of period-1 income. The coefficient-vector η_i specifies the commodity requirements per unit of investment, and the vector-valued function g_i measures the impact of investment on next year's endowments. This impact may depend on the random event. The symbol ϑ (r-vector) is the commodity basket in which transfers are expressed.

Prices are taken as given by the actors. They follow from market clearing, in period 1 and for each random event in period 2:

$$\begin{aligned} p_1 \geq 0 \quad & \perp \quad \Sigma_i x_{i1} + \Sigma_i \eta_i d_{i1} \leq \Sigma_i \overline{\omega}_{i1} \\ p_2(\varepsilon_2) \geq 0 \quad & \perp \quad \Sigma_i x_{i2}(\varepsilon_2) + \Sigma_i \eta_i \overline{d}_{i2} \leq \Sigma_i \omega_{i2} \end{aligned} \qquad (2.2)[2]$$

The model is completed with a price normalization rule (affecting prices in both periods simultaneously). Such a rule does not influence the real side due to the absence of non-homogeneities. The rule adopted here sets the price of the commodity basket in the first period 1 at one:

$$p_1 \vartheta = 1 \qquad (2.3)$$

The full model consists of the equations (2.1), (2.2) and (2.3). The instantaneous utility functions u_{it} are assumed to be strictly concave, continuously differentiable and nonsatiated (i.e. nondecreasing in all commodities and increasing in at least one of them), whereas for each commodity there is at least one actor with increasing utility. We also assume that the investment impact functions g_i are strictly concave and continuously differentiable, that the support (i.e. the area with positive values) of the subjective density factors $f_i^*(\varepsilon_2)$ is the same for all actors and that the endowments are such that feasible allocations exist. What can be said about existence and uniqueness of the solution?

Suppose that the number of possible events (states) is finite. Then the model has a standard AGE structure, with time and state considered as commodity characteristics. The existence of equilibria for models of this kind has been proved under alternative sets of assumptions, starting with Arrow and Debreu (1954). In the comprehensive analysis of the structure of general equilibrium models by Ginsburgh and Keyzer (1997) existence is shown on the basis of four properties of excess demand functions: single-valuedness, continuity, homogeneity of degree zero

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[2] The symbol $\perp$ indicates complementarity: $x \geq 0 \perp y \geq 0$ for the vectors x and y means that $x \geq 0, y \geq 0$ and $xy = 0$.

and compliance with Walras' law.[3] To obtain these properties they impose the following conditions:[4] (i) the production set has the possibility of inaction, is compact and strictly convex, (ii) the utility function is continuous, strictly concave and increasing, and (iii) the endowment vectors of the actors are non-negative with at least one positive element. Their proof applies also to our case since investment is in fact a production activity with input and output in different periods, whereas the liquidity constraints do not disturb the properties of the excess-demand functions (provided that they do not impede a feasible allocation).

Uniqueness of the solution can theoretically not be guaranteed, as never in an AGE model. The number of solutions depends on the multiplicity of the price solution of excess-demand system (2.2), since the solution of utility programme (2.1) has a one-to-one link with the prices, due to the strict concavity of its objective. Ginsburgh and Keyzer argue that under two additional properties of the excess-demand functions, i.e. continuous differentiability and desirability, the equilibria are finite in number and locally unique. The property of desirability (zero price implies positive excess demand) is automatically implied by the conditions listed above. But the property of continuous differentiability is disturbed by the presence of the liquidity constraints, which are not necessarily binding.

If the number of possible states is infinite (be it denumerable or continuum infinite), the existence proof requires generalized theorems that are applicable to an infinite-dimensional commodity space. Mas-Colell and Zame (1991) discuss the specific mathematical problems in going from a finite-dimensional to an infinite-dimensional space,[5] and provide an overview of existence theorems that cope with these problems. Unfortunately, the conditions in these theorems are rather abstract and not easy to verify. Furthermore, local uniqueness is largely unexplored in the infinite-dimensional setting. Mas-Colell and Zame mention the Negishi approach as a possible alternative that avoids most of the difficulties by casting the equilibrium problem in terms of welfare weights of the actors, which constitute a finite-dimensional space. This is indeed the approach followed by Ermoliev and Keyzer (1997). We will elaborate on it in the next section.

But one can anyhow assume a pragmatic attitude towards the existence issue, since the concavity of utility and investment impact functions makes it possible to formulate conditions for the solution of program (2.1), that are both necessary and sufficient. These conditions are derived in annex B, using a theorem of Hestenes as formulated in Takayama (1974). They consist of complementarity conditions of the constraints, i.e.

$$V_{i1} + \int V_{i2}(\varepsilon_2) f(\varepsilon_2) d\varepsilon_2 \leq 0 \quad \perp \quad \lambda_i \geq 0 \tag{2.4a}$$

$$V_{i1} \leq Z_{i1} \quad \perp \quad \mu_{i1} \geq 0 \tag{2.4b}$$

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[3] Ginsburgh and Keyzer formulate their assumptions on the excess demand functions along the lines of Arrow and Hahn (1971).

[4] The conditions mentioned here are from chapter 1, proposition 1.4. Some of them are relaxed in later chapters.

[5] For instance, compactness is more difficult to prove.

$$V_{i2}(\varepsilon_2) \le Z_{i2} - V_{i1} \quad \perp \quad \mu_{i2}(\varepsilon_2) \ge 0 \quad \text{for all } \varepsilon_2 \tag{2.4c}$$

and complementarity conditions of the activities, i.e.[6]

$$\frac{\partial u_{i1}/\partial x_{ik1}}{p_{k1}} \le \lambda_i + \mu_{i1} + \int \mu_{i2}(\varepsilon_2)\, f(\varepsilon_2)\, d\varepsilon_2 \quad \perp \quad x_{ik1} \ge 0 \tag{2.4d}$$

$$\frac{\partial u_{i2}/\partial x_{ik2}(\varepsilon_2)}{p_{k2}(\varepsilon_2)} f_i(\varepsilon_2 | I_1) \le \lambda_i\, f(\varepsilon_2) + \mu_{i2}(\varepsilon_2)\, f(\varepsilon_2) \quad \perp \quad x_{ik2}(\varepsilon_2) \ge 0 \quad \text{for all } \varepsilon_2 \tag{2.4e}$$

$$\int (\lambda_i + \mu_{i2}(\varepsilon_2))\, p_2(\varepsilon_2) \frac{\partial g_i(d_{i1}, \varepsilon_2)}{\partial d_{i1}} f(\varepsilon_2)\, d\varepsilon_2 \le p_1 \eta_i\, [\lambda_i + \mu_{i1} + \int \mu_{i2}(\varepsilon_2) f(\varepsilon_2) d\varepsilon_2] \quad \perp \quad d_{i1} \ge 0 \tag{2.4f}$$

Finding an equilibrium solution is equivalent to finding a feasible point of (2.2) – (2.4). Since unicity is not guaranteed theoretically, quantitative tests are necessary to check whether alternative solutions exist. One may observe that, in case of a finite number of states, conditions (2.4a) – (2.4f) are just the Kuhn-Tucker conditions[7] of program (2.1), known to be necessary and sufficient under concavity.

Summarizing, in this section we have identified necessary and sufficient conditions for the equilibrium solution of the model in excess-demand format. When the number of possible events in period 2 is finite, existence of the equilibrium can theoretically be proved. For the infinite case we will explore existence via the Negishi format.

[6] In fact, we define shadow price $\mu_{i2}(\varepsilon_2)$ as belonging to a version of the period-2 liquidity constraint in which left- and righthandside are multiplied by $f(\varepsilon_2)$. Hence, the factor $f(\varepsilon_2)$ after $\mu_{i2}(\varepsilon_2)$ in conditions (2.4d) - (2.4f).

[7] See e.g. Avriel (1976), theorems 4.38 and 4.39

Section 3
Negishi-format

The Negishi format consists of a two-period welfare programme at given welfare weights and a feedback component in which the welfare weights are adjusted in order to satisfy the budget constraints of the actors. The shadow prices of the commodity balances in the welfare program are the market prices, for period 1 and each event in period 2.

The computational advantage of the Negishi formulation over the excess-demand formulation comes from the iteration over deterministic variables (welfare weights) instead of stochastic ones (prices). In particular, decomposition of the intertemporal decision into two single-period programmes can be done more efficiently in an equilibrium search over deterministic welfare weights than in an equilibrium search over stochastic prices. The disadvantage of the Negishi formulation is the need to keep the price parameters in the liquidity constraints of the welfare programme equal to the shadow prices of the commodity balances, requiring an extra round of iterations. However, this disadvantage is definitely outweighed by the advantage.

The welfare program maximizes the social welfare function subject to the commodity balances of each period and event and to the liquidity constraints. At given positive welfare weights α, given period-1 price parameters $\tilde{p}_1$ and given liquidity bounds Z_1 and Z_2, the period-1 decision can be written as:

$$\max_{x_{i1}, d_{i1} \geq 0 \text{ for all } i} \quad \textstyle\sum_i \alpha_i u_{i1}(x_{i1}) + \int W(x_1, d_1; Z_2, \alpha, \tilde{p}_1, \varepsilon_2)\, f(\varepsilon_2) d\varepsilon_2 \qquad (3.1a)$$

$$\text{s.t.} \quad \textstyle\sum_i x_{i1} + \sum_i \eta_i d_{i1} \leq \sum_i \overline{\omega}_{i1} \qquad (p_1)$$

$$V_{i1} \leq Z_{i1} \quad \text{for all } i \qquad (\breve{\mu}_{i1})$$

$$\text{with} \quad V_{i1} = \tilde{p}_1(x_{i1} + \eta_i d_{i1}) - \tilde{p}_1(\overline{\omega}_{i1} + \bar{t}_{i1}\vartheta)$$

Shadow prices of the constraints are indicated between brackets. The function W specifies optimal period-2 welfare at given period-1 consumption and investment levels x_1 and d_1, given insurance bounds Z_2, given welfare weights α, given period-1 price parameters $\tilde{p}_1$ and given random event ε_2. It results from optimal allocation of consumption in period 2, subject to the commodity balances and liquidity constraints, and with utility premultiplied by the subjective density factors $f_i^*(\varepsilon_2) = f_i(\varepsilon_2 | I_1) / f(\varepsilon_2)$:

$$W(x_1, d_1; Z_2, \alpha, \tilde{p}_1, \varepsilon_2) =$$

$$\max_{x_{i2}(\varepsilon_2) \geq 0 \text{ for all } i} \quad \textstyle\sum_i \alpha_i f_i^*(\varepsilon_2)\, u_{i2}(x_{i2}(\varepsilon_2)) \qquad (3.1b)$$

s.t. $\sum_i x_{i2}(\varepsilon_2) + \sum_i \eta_i \bar{d}_{i2} \le \sum_i \omega_{i2}$ $\quad (p_2(\varepsilon_2))$

$V_{i2}(\varepsilon_2) \le Z_{i2} - V_{i1}$ $\quad (\breve{\mu}_{i2}(\varepsilon_2))$ for all i

with $V_{i1} = \tilde{p}_1(x_{i1} + \eta_i d_{i1}) - \tilde{p}_1(\bar{\omega}_{i1} + \bar{t}_{i1}\vartheta)$

$\omega_{i2} = \bar{\omega}_{i2}(\varepsilon_2) + g_i(d_{i1}, \varepsilon_2)$

$h_{i2}(\varepsilon_2) = \tilde{p}_2(\varepsilon_2)(\omega_{i2} + \bar{t}_{i2}\vartheta)$

$V_{i2}(\varepsilon_2) = \tilde{p}_2(\varepsilon_2)(x_{i2}(\varepsilon_2) + \eta_i \bar{d}_{i2}) - h_{i2}(\varepsilon_2)$

and with extra iterations such that $\tilde{p}_2(\varepsilon_2) = p_2(\varepsilon_2)$.

In the feedback component welfare weights α_i are adjusted such that

(i) budgets are satisfied (relative adjustment):

$$V_{i1} + \int V_{i2}(\varepsilon_2) f(\varepsilon_2) d\varepsilon_2 = 0 \quad \text{for all i} \tag{3.2}$$

(ii) the price of the commodity basket in period 1 equals one (normalization):

$$p_1\vartheta = 1 \tag{3.3}$$

whereas, furthermore, the period-1 price parameters are kept equal to the shadow prices and liquidity bounds are recalculated:

$$\tilde{p}_1 = p_1 \tag{3.4a}$$

$$h_{i1} = \tilde{p}_1(\bar{\omega}_{i1} + \bar{t}_{i1}\vartheta) \tag{3.4b}$$

$$Z_{i1} = \ell_{i1} h_{i1} \tag{3.4c}$$

$$Z_{i2} = \ell_{i2} h_{i1} \tag{3.4d}$$

The complete model in Negishi format is given by (3.1a), (3.1b), (3.2), (3.3) and (3.4). Below, we study existence of its solution, beginning with the case that the insurance constraints are not effective. The assumptions about the properties of the functions were formulated already in section 2. Neither program (3.1a) nor (3.1b) has functions as arguments which allows application of finite-dimensional optimization theorems.

The existence proof below uses similar arguments as the proof in section 3.1 of Ginsburgh and Keyzer (1997). Only solutions in which at least one of the welfare weights is positive, are relevant. Otherwise, the price normalization rule cannot hold. A priori it is not known whether all welfare weights will be positive. Therefore, uniqueness of allocations and differentiability of the function W cannot be assumed from the outset. As long as the latter has not been shown, the shadow prices in (3.1a) refer to the generalized Kuhn-Tucker saddlepoint inequality[8] instead of the standard Kuhn-Tucker theorem for differentiable functions.

Existence with non-binding insurance constraints

When the insurance constraints are non-binding, period-2 program (3.1b) can be analyzed at zero shadow prices $\breve{\mu}_{i2}(\varepsilon_2)$, whereas the feedback rule to update price parameters $\tilde{p}_2(\varepsilon_2)$ is not relevant for the solution. In this case, the period-2 program is independent of the period-1 consumption decision x_1. The maximum of (3.1b) exists (continuous objective on a compact feasible region). Due to the concavity and continuity of the objective and the constraints, properties of the maximum can be derived with the maximum theorem and perturbation theorem as formulated by Ginsburgh and Keyzer (1997).[9] By the maximum theorem, the function W is continuous in $(d_1, \alpha, \tilde{p}_1)$ and in ε_2, the optimal consumption variables $x_{i2}(\varepsilon_2)$ and shadow prices $p_2(\varepsilon_2)$ are uppersemicontinuous correspondences of $(d_1, \alpha, \tilde{p}_1)$, and the sets of optimal consumption variables and shadow prices are convex. By the perturbation theorem, the function W is concave in d_1.

Furthermore, since the program is convex, the generalized Kuhn-Tucker saddlepoint theorem shows that when $\hat{x}_2(\varepsilon_2)$ is a solution with corresponding shadow price $\hat{p}_2(\varepsilon_2)$, the following inequality holds for all $x_{i2}(\varepsilon_2) \geq 0$:

$$\Sigma_i \alpha_i f_i^*(\varepsilon_2) u_{i2}(x_{i2}(\varepsilon_2)) + \hat{p}_2(\varepsilon_2)[\Sigma_i \omega_{i2} - \Sigma_i x_{i2}(\varepsilon_2) - \Sigma_i \eta_i \bar{d}_{i2}] \quad \leq$$

$$\Sigma_i \alpha_i f_i^*(\varepsilon_2) u_{i2}(\hat{x}_{i2}(\varepsilon_2)) + \hat{p}_2(\varepsilon_2)[\Sigma_i \omega_{i2} - \Sigma_i \hat{x}_{i2}(\varepsilon_2) - \Sigma_i \eta_i \bar{d}_{i2}]$$

[8] See Avriel (1976), theorem 4.41.
[9] Appendix A, theorems A.3.1 and A.3.2.

By substituting $x_{i2}(\varepsilon_2) = 0$ for one of the actors and $x_{i2}(\varepsilon_2) = \hat{x}_{i2}(\varepsilon_2)$ for the other actors, one gets under the assumption that $u_{i2}(0) = 0$ [10] , for all i:

$$\alpha_i f_i^*(\varepsilon_2) u_{i2}(\hat{x}_{i2}(\varepsilon_2)) \geq \hat{p}_2(\varepsilon_2)\hat{x}_{i2}(\varepsilon_2) \tag{3.5}$$

Now we turn to (3.1a), the period-1 program. The continuity of W in ε_2 ensures that the integral in the objective exists.[11] Continuity and concavity are preserved under the integral operation. Since the objective is continuous in (x_1, d_1) and the feasible region is compact, the optimum exists. Due to the concavity of the utility function in x_1, the concavity of the integral in d_1 and the linearity of the constraints, the maximum theorem of a convex program can be applied again. It says that the optimal objective value of (3.1a) is continuous in $(\alpha, \tilde{p}_1)$, that the optimal variables (x_1, d_1) and shadow prices $(p_1, \breve{\mu}_1)$ are uppersemicontinuous correspondences of $(\alpha, \tilde{p}_1)$, and that the sets of optimal variables and shadow prices are convex. When $\hat{x}_1$ is an optimal consumption vector with corresponding shadow prices $\hat{p}_1, \breve{\mu}_1$, a similar argument as in (3.1b), on the basis of the generalized Kuhn-Tucker saddlepoint theorem, shows for all i:

$$\alpha_i u_{i1}(\hat{x}_{i1}) \geq \hat{p}_1 \hat{x}_{i1} + \breve{\mu}_{i1} \tilde{p}_1 \hat{x}_{i1} \tag{3.6}$$

Thus, programs (3.1a) and (3.1b) lead to solutions V_{i1} (i=1,..,m), $\int V_{i2}(\varepsilon_2) f(\varepsilon_2) d\varepsilon_2$ (i=1,…m) and p_1 which are uppersemicontinuous correspondences of $(\alpha, \tilde{p}_1)$. Furthermore, the sets of these three variables (respectively m-dimensional, m-dimensional and r-dimensional) are convex. What remains is the determination of values of α and $\tilde{p}_1$ that satisfy budgets (3.2), normalization rule (3.3) and feedback relations (3.4a). To this end, we denote the period-1 price correspondence as $\hat{p}_1(\alpha, \tilde{p}_1)$ and the budget deficit of actor i as $s_i(\alpha, \tilde{p}_1)$, and consider the set of equations

$$\alpha_i = \varphi \max\{\alpha_i - \rho s_i(\alpha, \tilde{p}_1), 0\} \quad \text{for } i=1,\dots,m \tag{3.7a}$$

$$\tilde{p}_{k1} = \hat{p}_{k1}(\alpha, \tilde{p}_1) \qquad \text{for } k=1,\dots,r \tag{3.7b}$$

in which ρ a positive constant, φ a normalization parameter such that $\sum_i \alpha_i = \overline{A}$ for some positive scalar $\overline{A}$, and

$$s_i(\alpha, \tilde{p}_1) = V_{i1} + \int V_{i2}(\varepsilon_2) f(\varepsilon_2) d\varepsilon_2 \,.$$

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[10] This assumption is not restrictive since one can always add a constant term to the utility function without changing the solution of the programme.

[11] See Apostol (1969), theorem 9-26 (for proper single integrals), section 10.6 (extension to multiple integrals) and theorem 14-2 (extension to improper integrals).

The righthandsides of (3.7a) and (3.7b) define a correspondence $G(\alpha, \tilde{p}_1)$ that we consider on the Cartesian product $C = A \times P$ with

$$A = \{\alpha \in R^m; \alpha \geq 0, \sum_i \alpha_i = \overline{A}\}$$

$P = \{\tilde{p}_1 \in R^r; \tilde{p}_1 \geq 0, \sum_k \tilde{p}_{k1} \leq \overline{P}\}$, in which $\overline{P}$ a positive scalar.

The operations in (3.7a) leave the property of uppersemicontinuity intact, whereas the (m-dimensional) set of outcomes of the righthandside remains convex. Therefore, G is an uppersemicontinuous correspondence from C into itself (provided that $\overline{P}$ is taken large enough), and $G(\alpha, \tilde{p}_1)$ is non-empty and convex for all $(\alpha, \tilde{p}_1) \in C$. Since, in addition, C is non-empty, compact and convex, Kakutani's fixed-point theorem can be applied to show that a solution $(\alpha, \tilde{p}_1)$ of system (3.7) indeed exists.

Scalar $\overline{A}$ can always be chosen such that normalization rule (3.3) is satisfied, i.e. $\hat{p}_1 \vartheta = 1$, due to the fact that $\hat{p}_1(\alpha, \tilde{p}_1)$ is linear homogeneous in α. As in any general equilibrium model without non-homogeneities, the normalization rule merely determines the absolute price level, without affecting commodity allocations and relative prices. What remains to be shown, is that $s_i = 0$ in the fixed point, for all actors i.

Suppose that $\alpha_i = 0$ for an actor i. Then, on one hand, (3.7a) implies that $s_i \geq 0$. But on the other hand, inequalities (3.5) and (3.6) show that consumption outlays of actor i are zero and, therefore, $s_i < 0$. The latter conclusion takes into account the optimal character of period-1 investments (which do not contribute negatively to the budget, due to possibility of inaction) and the assumptions made about the endowments. The contradiction means that in the fixed point all welfare weights are positive.

The final part of the proof is based on the observation that supply value equals demand value for each commodity in each period (following from the complementarity of shadow prices and constraints which is part of Avriel's formulation of the generalized Kuhn-Tucker saddlepoint theorem). Straightforward totaling of these balances gives $\sum_i s_i = 0$. Using this equality and knowing that the welfare weights are positive, summation of the equations of (3.7a) leads to the conclusion that $\varphi = 1$. Then, again from (3.7a), $s_i = 0$ for all i. Hence, the fixed point is indeed the solution of the model in Negishi format.

Existence with binding insurance constraints

With binding insurance constraints the period-2 feedback relation $\tilde{p}_2(\varepsilon_2) = p_2(\varepsilon_2)$ cannot be neglected and the existence proof of the period-2 solution needs a fixed-point theorem. Unfortunately, single-valuedness of the function W is not immediately clear then, which makes analysis of the period-1 solution rather complicated. We do not pursue this issue further.

Summarizing, without binding insurance constraints existence of the solution of the Negishi-format can be proved. All welfare weights turn out to be positive. With binding insurance constraints the situation is more difficult. We will come back to this case in the next section where equivalence with the excess demand format will be discussed.

Section 4
Equivalence of Negishi and excess-demand format

This section studies the equivalence of Negishi format and excess-demand format by formulating for programs (3.1a) and (3.1b) the Kuhn-Tucker conditions from the standard finite-dimensional optimization theory.[12] Unlike the Kuhn-Tucker saddlepoint theorem applied in the previous section, these conditions require differentiability of objective and constraints. For program (3.1b) differentiability simply holds due to the assumptions about the functions. For program (3.1a) it will be verified explicitly, making use of the property that all welfare weights are positive, at least when insurance constraints are not binding.

Kuhn-Tucker conditions

At given price parameters $\tilde{p}_2(\varepsilon_2)$, program (3.1b) is a convex program, regular in the Kuhn-Tucker sense, with differentiable functions. Therefore, the following conditions are necessary and sufficient for a solution:

$$\sum_i x_{i2}(\varepsilon_2) + \sum_i \eta_i \bar{d}_{i2} \le \sum_i \omega_{i2} \quad \perp \quad p_2(\varepsilon_2) \ge 0 \tag{4.1a}$$

$$V_{i2}(\varepsilon_2) \le Z_{i2} - V_{i1} \quad \perp \quad \breve{\mu}_{i2}(\varepsilon_2) \ge 0 \tag{4.1b}$$

and for i = 1,…,m :

$$\alpha_i f_i^*(\varepsilon_2) \frac{\partial u_{i2}}{\partial x_{i2}(\varepsilon_2)} - p_2(\varepsilon_2) - \tilde{p}_2(\varepsilon_2)\breve{\mu}_{i2}(\varepsilon_2) \le 0 \quad \perp \quad x_{i2}(\varepsilon_2) \ge 0 \tag{4.1c}$$

with $\omega_{i2}, V_{i2}(\varepsilon_2)$ and V_{i1} defined as in (3.1b).

To these conditions the period-2 feedback rule should be added:

$$\tilde{p}_2(\varepsilon_2) = p_2(\varepsilon_2) \tag{4.2}$$

We assume provisionally (to be verified later) that the period-2 optimum W is continuous in ε_2 and continuously differentiable and concave in (x_1, d_1). Continuity of W in ε_2 ensures that the integral in the objective of (3.1a) exists, as argued earlier in section 3. Due to the continuous differentiability of W in (x_1, d_1), the partial derivatives of the integral with respect to x_1 and d_1

[12] See e.g. Avriel (1976), theorems 4.38 and 4.39.

can be obtained by differentiation under the integral sign.[13] Concavity in (x_1, d_1) is preserved under the integral operation. The linear constraints of program (3.1a) guarantee Kuhn-Tucker regularity. Since in addition the utility function is concave and differentiable, the Kuhn-Tucker conditions are necessary and sufficient for the solution of (3.1a), at least under the provisional assumptions made above. They are given by:

$$\sum_i x_{i1} + \sum_i \eta_i d_{i1} \le \sum_i \overline{\omega}_{i1} \quad \perp \quad p_1 \ge 0 \tag{4.3a}$$

and for i = 1,…,m :

$$\alpha_i \frac{\partial u_{i1}}{\partial x_{i1}} + \int \frac{\partial W(x_1, d_1; Z_2, \alpha, \tilde{p}_1, \varepsilon_2)}{\partial x_{i1}} f(\varepsilon_2)\, d\varepsilon_2 - p_1 - \breve{\mu}_{i1}\tilde{p}_1 \le 0 \quad \perp \quad x_{i1} \ge 0 \tag{4.3b}$$

$$\int \frac{\partial W(x_1, d_1; Z_2, \alpha, \tilde{p}_1, \varepsilon_2)}{\partial d_{i1}} f(\varepsilon_2)\, d\varepsilon_2 - p_1\eta_i - \breve{\mu}_{i1}\tilde{p}_1\eta_i \le 0 \quad \perp \quad d_{i1} \ge 0 \tag{4.3c}$$

$$V_{i1} \le Z_{i1} \quad \perp \quad \breve{\mu}_{i1} \ge 0 \tag{4.3d}$$

with V_{i1} defined as in (3.1a).

Thus, the solution of the Negishi-format is characterized by conditions (3.2) – (3.4), (4.1) – (4.3) and by the marginal derivatives of W with respect to x_{i1} and d_{i1}. When the latter are equal to

$$\frac{\partial W(x_1, d_1; Z_2, \alpha, \tilde{p}_1, \varepsilon_2)}{\partial x_{ik1}} = -\breve{\mu}_{i2}(\varepsilon_2)\,\tilde{p}_{k1} \tag{4.4a}$$

$$\frac{\partial W(x_1, d_1; Z_2, \alpha, \tilde{p}_1, \varepsilon_2)}{\partial d_{i1}} = (1 + \breve{\mu}_{i2}(\varepsilon_2))\, p_2(\varepsilon_2) \frac{\partial g_i(d_{i1}, \varepsilon_2)}{\partial d_{i1}} - \breve{\mu}_{i2}(\varepsilon_2)\tilde{p}_1\eta_i \tag{4.4b}$$

the equivalence with the solution of the excess-demand format, i.e. conditions (2.2) – (2.4), can readily be seen, by substituting

$$\alpha_i = 1/\lambda_i,\ \breve{\mu}_{i1} = \mu_{i1}/\lambda_i,\ \breve{\mu}_{i2}(\varepsilon_2) = \mu_{i2}(\varepsilon_2)/\lambda_i\,.$$

We will study the equivalence separately for the case without and the case with binding insurance constraints.

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[13] See Apostol (1969), theorems 9-37 (for proper single integrals), section 10.6 (extension to multiple integrals) and theorem 14-24 (extension to improper integrals).

Equivalence with non-binding insurance constraints

In this case the solution of the Negishi format exists and all welfare weights are positive, as seen in the previous section. Therefore, the objective of (3.1b) is strictly concave in all its arguments which makes the optimal consumption allocation unique. It is the only allocation that satisfies the necessary and sufficient Kuhn-Tucker conditions (4.1). Uniqueness of the shadow prices $p_2(\varepsilon_2)$ follows directly from the uniqueness of the optimal vectors $x_{i2}(\varepsilon_2)$, by equations (4.1c). None of these prices is zero since the welfare weights are positive and since for each commodity there is at least one actor with increasing utility. Hence, all commodity balances are binding in the optimum.

Due to the continuous differentiability of the functions in (3.1b) and the uniqueness of its optimal variables and shadow prices, the function W is continuously differentiable in (x_1, d_1), according to the envelop theorem.[14] Since we know already from section 3 that W is continuous in ε_2 and concave in (x_1, d_1), all provisional assumptions made above in formulating the period-1 Kuhn-Tucker conditions are indeed satisfied. The envelope theorem also shows that the marginal derivatives of W are consistent with equations (4.4).[15] Hence, without binding insurance constraints the excess-demand format and Negishi-format are equivalent.

The multiplicity of the equilibrium solution can be characterized by further analysis of (3.1a). The function W is not only concave in d_1, but even strictly concave since the elements of the vector functions g_i are strictly concave and the commodity balances in period 2 are binding. Due to this strict concavity of W in d_1 (preserved under the integral operation), the strict concavity of the utility functions and the positive welfare weights, the optimal solution (x_1, d_1) is unique. Furthermore, the shadow prices p_1 and $\breve{\mu}_1$ are unique if the matrix of their coefficients in the binding constraints of (4.3b) and (4.3c) has full rank. This condition is indeed fulfilled since not all borrowing constraints can be binding at the same time (given that $Z_{i1} > 0$ for all i).

Hence, the solutions of $x_{i1}, d_{i1}, p_1, V_{i1}$ and $\int V_{i2}(\varepsilon_2) f(\varepsilon_2) d\varepsilon_2$ are single-valued continuous functions of $(\alpha, \tilde{p}_1)$. It means that the multiplicity of the equilibrium fully depends on the multiplicity of the fixed point $(\alpha, \tilde{p}_1)$ in solving system (3.7).

Equivalence with binding insurance constraints

In this case, it is difficult to validate the assumptions made in formulating the period-1 Kuhn-Tucker conditions above. But, more seriously, the marginal derivatives of W do not coincide with (4.4) due to feedback relations (4.2) that must be taken into account in applying the envelop theorem. They cause the following additional terms in the righthandside of (4.4a) respectively (4.4b): [16]

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[14] See the formulation and discussion in Ginsburgh and Keyzer (1997), appendix A, section 3.

[15] With $\breve{\mu}_{i2}(\varepsilon_2) = 0$ for all i.

[16] See annex C.

$$+\sum_j \breve{\mu}_{j2}(\varepsilon_2)\frac{\partial p_2(\varepsilon_2)}{\partial x_{ik1}} z_{j2} \tag{4.5a}$$

$$+\sum_j \breve{\mu}_{j2}(\varepsilon_2)\frac{\partial p_2(\varepsilon_2)}{\partial d_{i1}} z_{j2} \tag{4.5b}$$

with $z_{j2} = \overline{\omega}_{j2}(\varepsilon_2) + g_j(d_{j1}, \varepsilon_2) + \bar{t}_{j2}\vartheta - x_{j2}(\varepsilon_2) - \eta_j \overline{d}_{j2}$.

These additional terms imply that the Negishi-format is not equivalent to the excess-demand format in case of binding insurance constraints. To obtain equivalence, a correction would be necessary. But even now, (3.2) – (3.4) and (4.1) – (4.3) can be used to obtain the solution of the excess-demand format, provided that the derivatives of W are calculated via (4.4a) and (4.4b).

Therefore, regardless of the insurance constraints, the algorithm will operate on the basis of conditions (3.2) – (3.4), or rather (3.7), and (4.1) – (4.4). This format suggests a solution method with three loops: an outer loop over α and $\tilde{p}_1$, a middle loop over x_1 and d_1, and an inner loop over $x_{i2}(\varepsilon_2)$ and $\tilde{p}_2(\varepsilon_2)$. Ermoliev and Keyzer argue that the inner loop does not require convergence of the sequence of Monte-Carlo drawings at each value of (x_1, d_1). Instead, they propose to use a Stochastic Quasi-Gradient method (SQG method) which allows simultaneous updates of (x_1, d_1) and random drawings in the inner loop.

Summarizing, with non-binding borrowing and insurance constraints the Negishi-format is equivalent to the excess-demand format and has a solution of which the multiplicity depends directly on the multiplicity of the welfare weights. When borrowing constraints are binding, the same applies but the multiplicity is slightly more complicated, due to the period-1 shadow prices. When insurance constraints are binding, the Negishi-format is not equivalent to the excess demand-format and existence of the solution cannot be proved theoretically. In either of the three cases the equilibrium solution algorithm can be based on equations (3.7) and (4.1) – (4.4). We will discuss general algorithmic aspects in the next section, and continue with the concrete application in section 6.

Section 5
Algorithmic background

Here, we provide background information for the implementation of the algorithm in the next sections. We follow the usual distinction of initial point, search direction, step size and convergence criterion as the four main elements of an algorithmic process for solving an optimization problem or, alternatively, the corresponding set of first-order conditions. If one consider an n-dimensional space with vectors x (elements x_i), a subset X and the problem

$$\max_{x \in X} f(x)$$

then step k of the algorithmic search process is written as

$$x_i^{k+1} = x_i^k + \rho_i^k s_i^k \qquad k=0,1,2,\ldots$$

starting from initial point $x^0 = \bar{x}^0$ and continuing until the difference between x^{k+1} and x^k becomes smaller than a predetermined convergence parameter.

with ρ^k vector of step sizes in step k, with elements ρ_i^k
s^k search direction in step k, with elements s_i^k

For practical purposes, the step sizes are often considered as the product of a basic vector and a scalar relaxation parameter.

Deterministic methods

The theory of convergence of deterministic algorithms, as explained in Ortega and Rheinboldt (1970),[17] focuses on ascent methods, i.e. methods for which, provided that stepsizes ρ_i^k are sufficiently small:

$$f(x^{k+1}) \geq f(x^k) \qquad \text{for } k=0,1,\ldots..$$

Gradient methods, with $s_i^k = \frac{\partial f}{\partial x_i}(x^k)$ for all i, are ascent methods. With respect to the standard Euclidean norm they provide, locally, the steepest ascent. Also all methods in which the search

[17] Chapter 14. In fact, they discuss a minimization problem (and hence, descent methods).

direction s_i^k has a positive inner product with the gradient, i.e. $\sum_i s_i^k \frac{\partial f}{\partial x_i}(x^k) > 0$, are ascent methods. If the function f is bounded, an ascent method guarantees convergence of the sequence $\{f(x^k)\}$.

In order to prove that the algorithm indeed converges to a solution of the first-order conditions, one should also know that (i) $\lim_{k \to \infty} x^k = \hat{x}$ for some $\hat{x}$, and (ii) $\frac{\partial f}{\partial x_i}(\hat{x}) = 0$, assuming that the optimal point is an interior point of X. These conditions require additional constraints, on step size and search direction separately. The constraints on the step size must guarantee sufficient decrease in each step. The constraints on the search directions imply, in most cases, that they must be gradient-related which is stronger than just having a positive inner product with the gradient.[18] Apart from these constraints, there are conditions on the relation between the initial point and the point of convergence but they are rather mild and lead to 'semi-local' convergence (as opposed to local convergence in which the initial point must be close to the point of convergence).

In the application below we will not formally check the convergence conditions but follow a rather pragmatic approach and confine ourselves to selecting a combination of initial point, step sizes and search directions that works satisfactorily. We will focus on gradient methods.

Convergence theory does not pay much attention to the constraints that determine the feasible region X, at least not in a general setting. Implicitly, it is assumed that constraints can be dealt with in some way or another, e.g. by projecting each new point x^{k+1} onto the feasible area (one-dimensionally or orthogonally), by extending the objective function with penalty terms or by reformulating the problem as an unconstrained search for the saddlepoints of the Lagrangean. In the latter option, if $X = \{x \geq 0 \mid h(x) \leq b\}$, one considers the saddlepoint problem

$$\min_{\mu \geq 0} \max_{x \geq 0} \; f(x) + \mu[b - h(x)]$$

Avriel (1976)[19] shows that this problem is equivalent to the constrained problem, at least for a convex programme and provided that the Kuhn-Tucker regularity conditions are satisfied. We will apply both the projection and the saddlepoint approach.

Certain type of optimization problems may be subject to a feedback relation, i.e. have parameters in objective or constraints which depend on the optimal outcome or the corresponding shadow price of the optimization problem. In formula:

[18] For gradient-related methods scalars $c, k_0 > 0$ exist such that $\sum_i s_i^k \frac{\partial f}{\partial x_i}(x^k) \geq c \, \| s^k \| \, \| \frac{\partial f}{\partial x}(x^k) \|$ for all $k \geq k_0$. See Ortega and Rheinboldt, section 14.1.

[19] Section 3.3.

$$\max_{x \geq 0} f(x,z)$$

$$\text{s.t} \quad h(x,z) \leq b \quad (\mu)$$

with feedback relations $z = z(x,\mu)$. The optimization problem and feedback rule must be solved iteratively, until a solution $(\hat{x},\hat{\mu},\hat{z})$ is obtained which satisfies both. Feedback parameters may represent e.g. price dependencies of the coefficients of the functions f and h. They may also result from approximations of nonlinear functions or refer to outcomes of subproblems that are solved separately.

In our case, we have feedback relations for the welfare weights of the objective function of problem (3.1a) and for the price parameters in the liquidity constraints of problems (3.1a) and (3.1b). In formulating the iterative feedback process, we will rely on three practical guidelines:

i) solutions of both optimization programme and feedback rule must be locally unique and continuous,

ii) diverging tendencies should be avoided, i.e. when x or μ depends positively on z then z should depend negatively on x or μ,

iii) updates of feedback parameters should not be too abrupt.

Stochastic methods

Deterministic methods can no longer be applied when it is cumbersome to calculate the function exactly in each step, e.g. when the objective is an expected value:

$$f(x) = E\, g(x,\varepsilon) \quad \text{with } \varepsilon \text{ a random variable.}$$

Although, in principle, $f(x)$ and its gradient can be evaluated by means of Monte-Carlo drawings, this method is very time-consuming as part of an iterative process. In these cases the SQG-method of stochastic optimization provides a powerful alternative. As described in Ermoliev (2001), its main idea to use statistical (biased or unbiased) estimates of the gradient of the objective as search direction, e.g.

$$s_i^k = \frac{\partial g}{\partial x_i}(x^k,\varepsilon^k) \quad \text{in which } \varepsilon^k \text{ a random outcome.}$$

Such a search direction is called a stochastic quasigradient, hence the acronym SQG. One may also take more random outcomes and calculate the average derivative. The bias compared to the real gradient is the difference.[20]

[20] If the expected value of the bias is zero, the search directions may also be called stochastic gradients.

$$\frac{\partial\, E\, g(x,\varepsilon)}{\partial\, x_i} - \frac{\partial g}{\partial x_i}(x^k,\varepsilon^k).$$

The stochastic quasigradient is much easier to calculate than (in each step again) the gradient of the objective function. The power of the method is that it can be shown to converge under rather mild conditions. Convergence is defined as $\lim_{k\to\infty} x^k = \hat{x}$ with probability one, in which $\hat{x}$ in the set of optimal solutions of $\max_{x\in X} E\, g(x,\varepsilon)$.

The convergence conditions are discussed in Ermoliev (2001) and, more formally, in Ermoliev (1988). Broadly speaking, the function $f(x)$ must be concave and continuous, and the set X must be convex. The stepsizes are initially free but from a certain k_0 onwards: $\rho_i^k = 1/k$ or $\rho_i^k = 1/(k+1-k_0)$. The density function of the random variable should not depend on x.[21] Convergence holds at the same time for the objective and its gradient:

$$\lim_{n\to\infty} (1/n) \sum_{k=1}^{n} g(x^k,\varepsilon^k) = E\, g(\hat{x},\varepsilon)$$

and $$\lim_{n\to\infty} (1/n) \sum_{k=1}^{n} \frac{\partial g}{\partial x_i}(x^k,\varepsilon^k) = E \frac{\partial\, g(\hat{x},\varepsilon)}{\partial\, x_i}.$$

Furthermore, the convergence result can be extended to continuous functions $d(x,\varepsilon)$ allowing, among other things, calculation of statistical moments:

$$\lim_{n\to\infty} (1/n) \sum_{k=1}^{n} d(x^k,\varepsilon^k) = E\, d(\hat{x},\varepsilon).$$

SQG is not an ascent method, whatever stepsize one takes. Hence, convergence is not monotonous. In applications, averaging of successive stochastic quasi-gradients may be crucial for obtaining convergence.

So far, the feasible area X is assumed to be a simple set which can be dealt with in each iterative step via a projection. More complex constraints, depending possibly also on the random event, may be treated similarly as in the deterministic case, by a penalty or saddlepoint approach. In the latter method, under concavity of $f(x)$, convergence requires that the ratios of the stepsizes of x to the stepsizes of the multipliers tend to zero. If $f(x)$ is strictly concave, these ratios can also be one.[22]

[21] If the density function depends on x, hence if the probabilities are endogenous, the formulation of the SQG should be extended. See the section on applications in Ermoliev (2001).

[22] See the section on general theory in Ermoliev (2001).

SQG is applicable in a two-period setting in which the period-1 decision must be taken before the outcome of random event ε whereas the period-2 decision is taken afterwards.[23] In this case, the function g may be written as

$$g(x,\varepsilon) = h(x,\hat{y},\varepsilon) \quad \text{with} \quad \hat{y} = \arg\max_{y \in Y} h(x,y,\varepsilon)$$

in which y is the period-2 decision variable, with feasible region Y, and h the objective value for each combination of period-1 decision, period-2 decision and random outcome. When Y is given by constraints $v(x,y,\varepsilon) \geq 0$, the envelop theorem leads to the following form of the stochastic gradient:

$$s_i^k = \frac{\partial h}{\partial x_i}(x^k, \hat{y}^k, \varepsilon^k) + \hat{\lambda}^k \frac{\partial v}{\partial x_i}(x^k, \hat{y}^k, \varepsilon^k)$$

in which $\hat{\lambda}$ is the shadow price belonging to $\hat{y}$. Under convexity of the feasible region Y and concavity (in x and y) of the functions h and v, the earlier convergence results extend to the two-period setting.[24] In this way SQG will be applied in the next section, combined with constraints on period-1 variable x.

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[23] Such a setting, in which some decisions are taken before knowing the random event and other decisions afterwards, is indicated as a 'recourse problem' in stochastic optimization.

[24] See the section on two-stage stochastic programming in Ermoliev (2001).

Section 6
Outline of the algorithm

This section describes the three loops of the algorithm and their interaction:

The inner loop solves (3.1b):
determination of $x_2(\varepsilon_2)$, $p_2(\varepsilon_2)$ and $\breve{\mu}_2(\varepsilon_2)$, at given $\alpha, x_1, d_1, \tilde{p}_1, Z_2$ and ε_2.
The middle loop solves (3.1a):
determination of x_1, d_1, p_1 and $\breve{\mu}_1$, at given α, Z_1, Z_2 and $\tilde{p}_1$.
The outer loop solves (3.2) – (3.4):
determination of α and $\tilde{p}_1$ (Z_1 and Z_2 follow directly from $\tilde{p}_1$) .

Inner loop

The inner loop is a search process over prices, consumption and shadow prices of the insurance constraints. The search directions are based on Kuhn-Tucker conditions (4.1a) – (4.1c). They can be seen as gradients of the equivalent unconstrained saddlepoint problem:

$$\min_{p_2(\varepsilon_2),\breve{\mu}_{i2}(\varepsilon_2)\geq 0} \max_{x_{i2}(\varepsilon_2)\geq 0} \sum_i \alpha_i f_i^*(\varepsilon_2) u_{i2}(x_{i2}(\varepsilon_2)) + \tag{6.1}$$

$$p_2(\varepsilon_2)[\sum_i \omega_{i2} - \sum_i x_{i2}(\varepsilon_2) - \sum_i \eta_i \bar{d}_{i2}] +$$

$$\sum_i \breve{\mu}_{i2}(\varepsilon_2)[Z_{i2} - V_{i1} - V_{i2}(\varepsilon_2)]$$

$$\text{with} \quad V_{i2}(\varepsilon_2) = \tilde{p}_2(\varepsilon_2)(x_{i2}(\varepsilon_2) + \eta_i \bar{d}_{i2} - \omega_{i2} - \bar{t}_{i2}\vartheta) \ .$$

From starting points $x_{i2}^0(\varepsilon_2)$, $p_{k2}^0(\varepsilon_2)$ and $\breve{\mu}_{i2}^0(\varepsilon_2)$ onwards, the algorithm proceeds in the following way with iteration steps $\ell = 0,1,2,\ldots$, search directions s and (positive) step sizes ρ:

$$x_{ik2}^{\ell+1}(\varepsilon_2) = \max\{x_{ik2}^{\ell}(\varepsilon_2) + \rho_{ik2}^{x\ell} s_{ik2}^{x\ell}, 0\} \tag{6.2a}$$

$$p_{k2}^{\ell+1}(\varepsilon_2) = \max\{p_{k2}^{\ell}(\varepsilon_2) - \rho_{k2}^{p\ell} s_{k2}^{p\ell}, 0\} \tag{6.2b}$$

$$\breve{\mu}_{i2}^{\ell+1}(\varepsilon_2) = \max\{\breve{\mu}_{i2}^{\ell}(\varepsilon_2) - \rho_{i2}^{\mu\ell} s_{i2}^{\mu\ell}, 0\} \tag{6.2c}$$

with $$s^{x\ell}_{ik2} = \alpha_i f_i^{\,*}(\varepsilon_2)\frac{\partial u_{i2}}{\partial x_{ik2}(\varepsilon_2)} - p^{\ell}_{k2}(\varepsilon_2) - \breve{\mu}^{\ell}_{i2}(\varepsilon_2)\,\tilde{p}^{\ell}_{k2}(\varepsilon_2) \tag{6.3a}$$

$$s^{p\ell}_{k2} = \Sigma_i\,\omega_{ik2} - \Sigma_i\, x^{\ell}_{ik2}(\varepsilon_2) - \Sigma_i\,\eta_{ik}\bar{d}_{i2} \tag{6.3b}$$

$$s^{\mu\ell}_{i2} = Z_{i2} - V_{i1} - \tilde{p}^{\ell}_{2}(\varepsilon_2)(x^{\ell}_{i2}(\varepsilon_2) + \eta_i\bar{d}_{i2} - \omega_{i2} - \bar{t}_{i2}\vartheta) \tag{6.3c}$$

The search directions are the gradients of the objective function of (6.1) for variables under the maximand, and the opposite of the gradients for variables under the minimand. The updates are performed simultaneously (Jacobi-iterations). The equality between $p_2(\varepsilon_2)$ and $\tilde{p}_2(\varepsilon_2)$ can directly be implemented in (6.3a) – (6.3b). Hence, there is no need for a separate iteration over $\tilde{p}^{\ell}_{k2}(\varepsilon_2)$. Non-negativity of the variables overrules the updates. When the process converges, Kuhn-Tucker conditions (4.1a) – (4.1c) and feedback rule (4.2) are automatically satisfied.

Since the inner loop has to be solved in each step of the other loops, an improvement in its functioning will speed up the algorithm considerably. Therefore, any analytical simplification allowed by the functional form, is applied and, evidently, all utility derivatives are calculated analytically.

Middle loop

The middle loop is a search process over prices, investment, consumption and shadow prices of the borrowing constraints. It is based on Kuhn-Tucker conditions (4.3a) – (4.3d). Again, it can be seen as gradient search in an equivalent unconstrained saddlepoint problem:

$$\min_{p_1 \geq 0, \breve{\mu}_{i1} \geq 0 \text{ for all } i} \quad \max_{x_{i1}, d_{i1} \geq 0 \text{ for all } i} \tag{6.4}$$

$$\Sigma_i\,\alpha_i u_{i1}(x_{i1}) + \int W(x_1, d_1; Z_2, \alpha, \tilde{p}_1, \varepsilon_2)\, f(\varepsilon_2)d\varepsilon_2 + p_1(\Sigma_i\,\overline{\omega}_{i1} - \Sigma_i\, x_{i1} - \Sigma_i\,\eta_i d_{i1}) + \Sigma_i\breve{\mu}_{i1}(Z_{i1} - V_{i1})$$

with $V_{i1} = \tilde{p}_1(x_{i1} + \eta_i d_{i1} - \overline{\omega}_{i1} - \bar{t}_{i1}\vartheta)$

The iteration steps are indicated by superscript r (r=0,1,2,..). The search directions with respect to p_1 and $\breve{\mu}_1$ are the opposite of the gradients of the objective function of (6.4):

$$s^{pr}_{ik1} = \Sigma_i\,\overline{\omega}_{ik1} - \Sigma_i\, x^{r}_{ik1} - \Sigma_i\,\eta_{ik} d^{r}_{i1} \tag{6.5a}$$

$$s^{\mu r}_{i1} = Z_{i1} - V^{r}_{i1} \tag{6.5b}$$

with $V_{i1}^{r} = \tilde{p}_1(x_{i1}^{r} + \eta_i d_{i1}^{r} - \overline{\omega}_{i1} - \bar{t}_{i1}\vartheta)$.

The search directions with respect to x_1 and d_1 are determined as stochastic quasi-gradients, implying that the derivatives of the function W are taken at a given drawing of ε_2 (in each iteration a new drawing, hence also the drawing has superscript r):

$$s_{ik1}^{xr} = \alpha_i \frac{\partial u_{i1}(x_{i1}^{r})}{\partial x_{ik1}} + \frac{\partial W(x_1^{r}, d_1^{r}; Z_2, \alpha, \tilde{p}_1, \varepsilon_2^{r})}{\partial x_{ik1}} - p_{k1}^{r} - \breve{\mu}_{i1}^{r}\tilde{p}_{k1} \qquad (6.6a)$$

$$s_{i1}^{dr} = \frac{\partial W(x_1^{r}, d_1^{r}; Z_2, \alpha, \tilde{p}_1, \varepsilon_2^{r})}{\partial d_{i1}} - p_1^{r}\eta_i - \breve{\mu}_{i1}^{r}\tilde{p}_1\eta_i \qquad (6.6b)$$

The derivatives of W are calculated as in (4.4a) and (4.4b). Step sizes are indicated analogously to search directions, with ρ instead of s. All step sizes are positive. Again, the updates are performed simultaneously and overruled by non-negativity of the variables:

$$x_{ik1}^{r+1} = \max\{x_{ik1}^{r} + \rho_{ik1}^{xr} s_{ik1}^{xr}, 0\} \qquad (6.7a)$$

$$d_{i1}^{r+1} = \max\{d_{i1}^{r} + \rho_{i1}^{dr} s_{i1}^{dr}, 0\} \qquad (6.7b)$$

$$p_{k1}^{r+1} = \max\{p_{k1}^{r} - \rho_{k1}^{pr} s_{k1}^{pr}, 0\} \qquad (6.7c)$$

$$\breve{\mu}_{i1}^{r+1} = \max\{\breve{\mu}_{i1}^{r} - \rho_{i1}^{\mu r} s_{i1}^{\mu r}, 0\} \qquad (6.7d)$$

The middle loop may be summarized as follows:

Select feasible initial values x_1^0, d_1^0, p_1^0 and $\breve{\mu}_1^0$. Then for r = 0,1,2,…

- perform random drawing ε_2^{r} from density $f(\varepsilon_2)$
- solve inner loop (period 2)
- determine search directions via (6.5) and (6.6)
- adjust variables via (6.7)
- stop when convergence is reached.

As discussed in section 5, in the point of convergence the expected derivatives of the objective function are obtained by averaging the derivatives in the successive drawings. Therefore, the Kuhn-Tucker conditions (4.3a) – (4.3d) are automatically satisfied in the point of convergence.

Outer loop

The outer loop is a feedback process over welfare weights and period-1 price parameters. The search process follows equations (3.7), with search directions based on budget and price gaps, respectively. Price normalization rule (3.3) is applied only after convergence has been obtained. During the iterations the simple requirement is imposed that the welfare weights add to a positive constant $\overline{A}$.

The iterative process starts from initial values α^0 and $\tilde{p}_1^0$ and proceeds with steps z = 0,1,2,…

$$\alpha_i^{z+1} = \max\{\alpha_i^z - \rho_i^{\alpha z} s_i^{\alpha z}, 0\} \tag{6.8a}$$

$$\tilde{p}_{k1}^{z+1} = \tilde{p}_{k1}^z + \rho_k^{pz} s_k^{pz} \tag{6.8b}$$

with after (6.8a) a projection (orthogonal or coordinatewise) of the welfare weights on the set

$$\{\alpha \in R^m \,|\, \alpha \geq 0, \textstyle\sum_i \alpha_i \leq \overline{A}\}.$$

Search directions are given by

$$s_i^{\alpha z} = V_{i1}^z + \int V_{i2}^z(\varepsilon_2) f(\varepsilon_2) d\varepsilon_2 \tag{6.9a}$$

$$s_k^{pz} = p_{k1}^z - \tilde{p}_{k1}^z \tag{6.9b}$$

in which the expected budget deficit and period-1 prices are calculated at $\alpha^z, \tilde{p}_1^z$.

Stepsizes $\rho_i^{\alpha z}$ and ρ_k^{pz} are positive. Hence, the welfare weights are reduced when the actor has a budget deficit and raised when the actor has a budget surplus. The stepsizes ρ_k^{pz} determine the degree of (positive) adjustment of the price parameters to the newly calculated period-1 prices. Since budget deficits are positively related to welfare weights and period-1 prices are negatively related to the period-1 price parameters in the borrowing constraints, the feedback process has no diverging tendencies. In the point of convergence, budget constraints (3.2) and price equality (3.4a) are automatically satisfied.

Section 7
Functional forms

In the current implementation of the algorithm, functional forms are rather standard. The instantaneous utility function is an iso-elastic transformation of a Stone-Geary function, preceded by an exogenous time preference parameter. The random sources are independently and uniformly distributed, with marginal density functions f^n. Endowment risk is specified by sector. For each sector it is linked by a pointer $\iota(k)$ to, at most, one of the random sources. The same random source may affect several sectors. Investment is specified as a production function with decreasing returns to scale that needs inputs in period 1 and yields output in period 2. Two versions are distinguished. In one version the investment impact is subject to the same random events as the exogenous endowments, in the other version the investment impact is non-stochastic. More specifically, the following functional forms are used:

$$\overline{\omega}_{ik2}(\varepsilon_2) = \overline{\omega}^{\circ}_{ik2}(1+\varepsilon_{n2}) \quad \text{with } n = \iota(k) \text{ and } \overline{\omega}^{\circ}_{ik2} \in R \tag{7.1}$$

$$u_{it}(x_{it}) = e^{-\delta(t-1)}\, v(w_i(x_{it});\sigma) \tag{7.2}$$

with
$$v(w;\sigma) = \begin{cases} (w^{1-1/\sigma}-1)/(1-1/\sigma) & \text{for } \sigma > 0, \sigma \neq 1 \\ \log(w) & \text{for } \sigma = 1^{25} \end{cases}$$

$$w_i(x_{it}) = \prod_k (x_{ikt} - \gamma_{ik})^{\beta_{ik}} \qquad x_{ikt} \geq \gamma_{ik} + s \quad \text{(all i,k,t)}$$

in which s a very small, positive value and $\delta, \beta_{ik}, \gamma_{ik} \geq 0$, $\sum_i \beta_{ik} > 0$ and $\sum_k \beta_{ik} = 1$

$$g_i(d_{i1},\varepsilon_2) = \zeta_i d_{i1}{}^{\nu}\, \overline{\omega}_{i2}(\varepsilon_2) \tag{7.3a}$$

$$\text{or} \quad \zeta_i d_{i1}{}^{\nu}\, \overline{\omega}^{\circ}_{i2} \tag{7.3b}$$

with $\overline{\omega}_{i2}(\varepsilon_2) \in R^r$, $\overline{\omega}^{\circ}_{i2} \in R^r$, $\zeta_i \in R$, $\nu \in R$, $\zeta_i > 0$ and $0 < \nu < 1$.

Stochastic specification:

$$f^n(\varepsilon_{n2}) = 1/(2\tau_n) \quad \text{for } \varepsilon_{n2} \in [-\tau_n, \tau_n] \quad \text{and } 0 \text{ elsewhere} \tag{7.4}$$

with $0 < \tau_n < 1$ and small enough to ensure that total endowments of each commodity are always sufficient to satisfy committed demand

[25] Then $v(w;1) = \lim_{\sigma \to 1} v(w;\sigma)$.

whereas $f(\varepsilon_2) = \prod_n f^n(\varepsilon_{n2})$.

Subjective density specification:

$$\frac{f_i^n(\varepsilon_{n2} | I_1)}{f^n(\varepsilon_{n2})} \begin{cases} = 1 + \psi_i & \text{for } 0 < \varepsilon_{n2} \le \tau_n \\ = 1 & \text{for } \varepsilon_{n2} = 0 \\ = 1 - \psi_i & \text{for } -\tau_n \le \varepsilon_{n2} < 0 \end{cases} \tag{7.5}$$

with $\psi_i \in (-1, 1)$ indicating the degree of optimism or pessimism of actor i,

$$\text{whereas } f_i^*(\varepsilon_2) = \frac{f_i(\varepsilon_2 | I_1)}{f(\varepsilon_2)} = \prod_n \frac{f_i^n(\varepsilon_{n2} | I_1)}{f^n(\varepsilon_{n2})}. \tag{7.6}$$

Utility function (7.2) is continuously differentiable, nonsatiated and strictly concave,[26] whereas the investment impact function (7.3) is continuously differentiable and strictly concave, for $d_{i1} > 0$. Therefore, the assumptions made in section 2 with respect to functional forms are satisfied. Equations (7.2) and (7.3) lead to the following derivatives:

$$\frac{\partial u_{it}(x_{it})}{\partial x_{ikt}} = e^{-\delta(t-1)} \frac{\beta_{ik}}{(x_{ikt} - \gamma_{ik})} w_i(x_{it})^{1-1/\sigma} \qquad \text{for } x_{ikt} \ge \gamma_{ik} + s \tag{7.7}$$

$$\frac{\partial g_i(d_{i1}, \varepsilon_2)}{\partial d_{i1}} = \nu \frac{g_i(d_{i1}, \varepsilon_2)}{d_{i1}} \qquad \text{for } d_{i1} > 0 \tag{7.8}$$

With derivatives (7.7) the process in the inner loop can be simplified, since substitution in conditions (4.1c) gives the following expression of $x_{i2}(\varepsilon_2)$ in terms of $p_2(\varepsilon_2)$ and $\breve{\mu}_{i2}(\varepsilon_2)$:

$$x_{ik2}(\varepsilon_2) - \gamma_{ik} = \tag{7.9}$$

$$\frac{\beta_{ik}\left(\alpha_i f_i^*(\varepsilon_2) e^{-\delta}\right)^{\sigma}}{p_{k2}(\varepsilon_2) + \breve{\mu}_{i2}(\varepsilon_2)\tilde{p}_{k2}(\varepsilon_2)} \left(\prod_{k'} \left(\frac{\beta_{ik'}}{p_{k'2}(\varepsilon_2) + \breve{\mu}_{i2}(\varepsilon_2)\tilde{p}_{k'2}(\varepsilon_2)} \right)^{\beta_{ik'}} \right)^{\sigma-1}$$

Hence, the inner loop reduces to a search process over $p_2(\varepsilon_2)$ and $\breve{\mu}_{i2}(\varepsilon_2)$, according to (6.2b) and (6.2c), with analytical updates of $x_{i2}(\varepsilon_2)$.

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[26] At least, when neglecting commodities k for which β_{ik} is zero.

For $\sigma = 1$ a further simplification is possible. Then the last factor on the righthandside of (7.9) vanishes and summation over i gives, after substitution of $\tilde{p}_{k2}(\varepsilon_2) = p_{k2}(\varepsilon_2)$ and commodity balances (4.1a), an analytical expression for the period-2 prices:

$$p_{k2}(\varepsilon_2) = \frac{\sum_i \beta_{ik} \alpha_i f_i^*(\varepsilon_2) e^{-\delta} / (1 + \breve{\mu}_{i2}(\varepsilon_2))}{\sum_i (\omega_{ik2} - \eta_{ik} \bar{d}_{i2} - \gamma_{ik})} \quad (7.10)$$

Hence, for $\sigma = 1$ the inner loop reduces to a search process over shadow prices $\breve{\mu}_{i2}(\varepsilon_2)$, according to (6.2c), with analytical updates of $x_{i2}(\varepsilon_2)$ and $p_2(\varepsilon_2)$.

Section 8
Implementation and experience

Here, we discuss the experience in applying the algorithm to an illustrative case with four actors, eight commodities and the functional forms of the previous section. The simulations and their economic interpretation are described in Van Veen (2001). The algorithm is programmed in Fortran 90, with double precision. The scheme with the major routines and their interaction is added as annex D.

First, some general remarks. As initial points we take interior points of the area of feasible allocations. The search directions of section 6 are followed strictly since they characterize the algorithm, but averaging of successive directions is allowed (and often crucial) in stochastic optimization. Step sizes and convergence criterions require considerable testing. Preferably, they should be defined relative to the size of the variables that are optimized. If step sizes are too large, the algorithm easily fails; if they are too small, the algorithm stagnates. The theoretical requirement (in stochastic optimization) of ultimately declining step sizes can be imposed at any moment in the iterative process. In applications it is quite possible that convergence is reached already before that moment. Underrelaxation (weighted average of old and new point) may be important to obtain convergence and, especially, to let the variables move towards a bound only gradually.

Below, we successively discuss the implementation of the algorithm in each of the loops, the overall performance, main lessons and issues for improvement.

Inner loop

A fully numeric search over consumption and prices is too slow, due to the high number of calls of the period-2 solution. Therefore, we use analytical expression (7.9) for $x_{i2}(\varepsilon_2)$, and if $\sigma = 1$ also analytical expression (7.10) for $p_2(\varepsilon_2)$. Hence, for $\sigma \neq 1$ the search process iterates over commodity prices and shadow prices of insurance, and for $\sigma = 1$ only over the latter.

The loop is called at each different random event. Using each time the same initial point for the commodity prices would be inefficient. Therefore, a priori categories of random outcomes are distinguished (for each commodity separately), each with its own price initialization. The categories are defined on the basis of endowment availability. The shadow prices of insurance are reinitialized each time at zero.

The search directions are as in (6.3b) and (6.3c). All step sizes are independent of iteration number ℓ. Price step sizes are commodity-specific. After price update (6.2b) underrelaxation is applied (hence, an additional lag in the adjustment) in order to prevent prices from going to zero too abruptly. In hindsight, such a lag can also be interpreted as a (uniformly applied) reduction of the step size but it is safer to impose it explicitly.

Convergence is based on first-order conditions (4.1a) and (4.1b). The commodity balance must be sufficiently close to zero (since in our case all prices must be positive). ‘Sufficiently close to zero’ is defined relative to total supply. The criterion for price convergence is applied rather leniently as long as the variables of the other loops are still changing, but convergence of the shadow price of insurance is kept strict from the beginning onwards. Non-convergence in period 2 is accepted (maximal number of iterations set at 1000) provided that it does not happen too often.

Middle loop

Initialization is only necessary at the very beginning, i.e. at the first application of the middle loop. Afterwards, each middle loop starts from the point of convergence of the previous one.

Search directions are as in (6.5a) – (6.5b) and (6.6a) – (6.6b). Step sizes are independent of iteration number r or, more precisely, the moment of their gradual reduction is not reached in the application. Price step sizes are commodity-specific and proportional to the price level. The actual consumption step sizes ρ_{ik1}^{xr} are made dependent on σ to take into account the influence of the latter on marginal utility, see (7.7), but not on actor i or on commodity k. The update of insurance bounds Z_{i2} is directly linked to the period-1 consumption and investment updates (following their definition).

The effect of the shadow price of the insurance constraint of period 2 on the SQG of period-1 consumption is applied with a considerable lag (factor only 0.0001), in order to temper the effects of individual events. In fact, it is a kind of averaging of the stochastic quasi-gradients.

Consumption is adjusted as in (6.7a) but taking into account functional form (7.2). Hence, the lower bound equals commitment $\gamma_{ik} + s$ (not necessarily zero). Underrelaxation is applied to prevent consumption from abruptly going to the committed lower level (which would result in huge derivatives).

Initially, full convergence of the middle loop is not required and the maximal number of iterations is kept limited. Only when the outer loop is well underway, strict convergence of the middle loop is imposed. The convergence criterion is based on simultaneous checks of first-order conditions (4.3a) – (4.3d), as well as on the changes in consumption, prices, investments and borrowing shadow prices compared to the previous iteration.

The investment update is, from a certain iteration number onwards, shifted upward to the outer level. In other words, then d_1 is kept fixed in the middle level whereas it is updated in the outer level based on the average SQG, i.e. the average outcome of equation (6.6b), computed in the previous middle level.

Outer loop

By definition, this loop has just one initialization. All initial welfare weights α taken equal. The same applies to the initial price parameters $\tilde{p}_1$.

The search directions $s_i^{\alpha z}$ are as in (6.9a), but with the budget gaps expressed relative to the total endowment value of the actor. Furthermore, a considerable lag is applied, averaging the search directions of successive iterations. The latter is done to reduce small-sample biases that may arise from an accidentally fast convergence of the middle loop (with only a limited number of random drawings for period 2). Search directions s_k^{pz} are as in (6.9b). Step sizes are independent of iteration number z. The updates of period-1 liquidity bounds Z_{i1} are not directly linked to parameters $\tilde{p}_1$, but follow their own pace.

Welfare weights are projected orthogonally onto the feasible area. Scaling would be an alternative, but appears to make not much difference. After projection of welfare weights, underrelaxation is applied.

As mentioned above, from a certain moment onwards the adjustment of d_1 is included in the outer loop, following averaged SQG's as search direction. Its step size is constant (not decreasing when the number of iterations gets larger), with underrelaxation to prevent investment from abruptly going to zero.

The convergence criterion is based on the size of the budget gaps, expressed relative to the total endowment value, and on the changes in investment and price levels compared to the previous iteration.

Overall performance

On the whole the algorithm works adequately for the simulations described in Van Veen (2001). Table 8.1 gives an overview of the number of iterations required in the various runs with widespread, aggregate risk. These runs have eight independent random shocks, viz. one for the endowments of each of the eight commodities. The term 'stochastic investment impact' refers to specification (7.3a), 'deterministic investment impact' to specification (7.3b). Optimism and pessimism are specified as in (7.5), with $\psi_i = 0.2$ respectively $\psi_i = -0.2$. By way of example, the technical model report (specification and convergence results) of one of the runs is added as annex E.

After convergence of the algorithm (i.e. ex-post), normalization rule (2.3) is applied and Monte-Carlo simulations of (3.1b) are performed at given welfare weights and given period-1 solution. For these Monte-Carlo simulations a sufficiently large number of drawings is taken (and a rather strict convergence criterion) in order to ensure that mean and variance of period-2 variables are properly calculated. We take 100,000 drawings (with maximally 5000 iterations to obtain convergence in period 2).

Table 8.1. Overview of number of iterations in simulation runs with widespread risk

Description of run	Inner loop (average)	Middle loop (average)	Outer loop
A. Rational expectations; investment impact deterministic; $\sigma = 1$; $\ell_{i1} = 0.05$; ℓ_{i2} large	1	204	10000
B. Rational expectations; investment impact stochastic; $\sigma = 1$; $\ell_{i1} = 0.05$; ℓ_{i2} large	1	205	1254
C. Rational expectations; investment impact deterministic; $\sigma = 0.75$; $\ell_{i1} = 0.05$; ℓ_{i2} large	65	164	2946
D. Rational expectations; investment impact deterministic; $\sigma = 1.50$; $\ell_{i1} = 0.05$; ℓ_{i2} large	79	268	5000
E. Correct expectations (poor optimistic); investment impact deterministic; $\sigma = 1$; $\ell_{i1} = 0.05$; ℓ_{i2} large	1	272	10000
F. Correct expectations (poor pessimistic); investment impact deterministic; $\sigma = 1$; $\ell_{i1} = 0.05$; ℓ_{i2} large	1	294	10000
G. Rational expectations; investment impact deterministic; $\sigma = 1$; $\ell_{i1} = 0$; ℓ_{i2} large	1	245	10000
H. Rational expectations; investment impact deterministic; $\sigma = 1$; $\ell_{i1} = 0.05$; $\ell_{i2} = 0.005$	9	308	2175
I. Correct expectations (poor optimistic); investment impact deterministic; $\sigma = 1$; $\ell_{i1} = 0.05$; $\ell_{i2} = 0.10$	29	878	5000
J. Correct expectations (poor pessimistic); investment impact deterministic; $\sigma = 1$; $\ell_{i1} = 0.05$; $\ell_{i2} = 0.10$	28	868	5000

When applied on a standard PC of vintage 2001 (254 MB RAM, 32-bit file system and virtual memory, 18.5 GB hard disk space, Pentium III processor with 1000 MHz) with Fortran 90 (using compiler MS Powerstation 4.0), the algorithm takes 1 minute for run B (fastest run of table 8.1, with 1 by 205 by 1254), including the ex-post calculations. Run D is the slowest run (79 by 268 by 5000), taking 44 minutes of which 5 minutes for ex-post. Furthermore, run I (29 by 878 by 5000) takes 26 minutes of which 2 minutes ex-post. It may seem surprising that the convergence process of run I is faster than convergence of run D but the inner loop of run I has iterations over shadow prices of liquidity, which are relatively simple compared to the commodity price iterations of the inner loop of D. Obviously, combining both types would increase the number of iterations. For instance, a run like H has on average 169 iterations in the inner loop for $\sigma = 0.75$ and $\ell_{i2} = 0.02$.

Uniqueness of the solution is tested by starting from different initial welfare weights, different initial period-1 investment levels and different initial seeds of the random process. In none of the runs multiplicity is found.[27]

Lessons

The following lessons may be derived from the process:

a) Attempts without the saddlepoint (min-max) approach appear to fail. In such attempts the commodity balances must be treated as constraints, to be met by projecting the solution of consumption and investment onto the feasible area, without a role for price adjustment. Then, consumption or investment volumes tend to come too close to their lower bounds (i.e. committed consumption or zero investment) leading to huge derivatives that cause the process to derail. Protection via lagged adjustments does not work. The saddlepoint method does not have this problem since it maintains the link between price adjustment and commodity balance.

b) Even with the saddlepoint approach, underrelaxation is important to let the variables move gradually and not approach the bounds too fast.

c) Persistent convergence problems appear to exist due to the interaction between price and investment updates in the middle loop. Therefore, at some stage the SQG's of investment have to be averaged by shifting the investment update upwards to the outer loop.

d) Price initialization via categories and commodity dependence of the step size prove to be extremely important in improving the speed of convergence of price iterations in the inner loop (relevant for $\sigma \neq 1$).

e) Averaging of the effect of the shadow price of the period-2 insurance constraints on the SQG's of period-1 is crucial to obtain convergence.

Issues for improvement

Table 8.1 mentions several times 5000 or 10000 iterations in the outer loop, a number that has been imposed as maximum. It appears that in these cases simultaneous adjustment of welfare weights and investments (even with considerable adjustment lags) leads to problems in meeting the formal convergence criterion although, in fact, the solution has been obtained. The latter can be seen from the first-order conditions of middle and outer loop which are OK in all runs above. Hence, the process tends to linger around the final point of convergence. A satisfactory 'final touch' to end the algorithm is still to be found.

[27] In run G the period-2 price level differs between alternative initializations. However, this is an issue of underdetermination rather than multiplicity, since in this specific case (without loans) there is no link between period-1 and period-2 prices. With just a little borrowing, the underdetermination disappears.

A second point that is not yet satisfactory, is the large number of iterations in the inner loop for $\sigma \neq 1$, and particularly the problem in finding the solution when the value of σ deviates more from 1 than in table 8.1, for instance when $\sigma = 0.5$ or $\sigma = 2$. In these cases the inner loop is taking too many iterations. Apparently, the specification of its price iterations is not yet general enough.

A third point for improvement is the speed and the accuracy of the middle loop under a combination of correct expectations and insurance bounds, like in runs I and J. In these runs, as opposed to the case of rational expectations, averaging of the shadow prices of the insurance constraints appears to be insufficient to guarantee smooth convergence of the SQG process.

Section 9
Evaluation

On the whole, the three-loop algorithm functions well. The deterministic inner loop for the period-2 allocation, the SQG middle loop for the period-1 allocation and the Negishi outer loop for the intertemporal budgets can indeed be fine-tuned so as to lead to the equilibrium solution. In the process it appears essential to represent commodity and liquidity constraints via a saddlepoint formulation (to maintain price-volume interactions) and to average some SQG search directions. But in the end, it is mainly the speed of the inner loop that determines the success of the algorithm. Equilibrium is found also in runs with binding insurance constraints, when its existence is not known a priori. Cases of multiple solutions (theoretically not to be excluded) are not found. The algorithm requires further improvement regarding its stopping criterion, to prevent cycling around the final solution, and its robustness, such that it can smoothly address a wider range of parameters and cases.

A logical next step would be the extension to more than two periods. Then the recursive character of the approach becomes more complicated, as in dynamic programming. In such a setting it is imperative that the solutions for the separate periods are rapidly obtained. Therefore, Ermoliev and Keyzer (1998) explain a dual approach in which one specifies a closed form of event-specific value functions and obtains the solutions of the separate periods just by differentiation. If one wishes to maintain the current (primal) functional forms for utility and investment, analytical approximation of the event-specific solutions of separate periods (in terms of variables of the previous periods, welfare weights and bounds) might be an alternative. In such a set-up the inner loops would be performed only at the initial stage. In the actual solution phase they would be replaced by the approximations. In both approaches the middle loop would still consist of the SQG-approach but now in a multi-period context, bearing in mind the moments of observing the successive random events, whereas the iteration over the welfare weights would remain the outer loop. Closure rules (e.g. if governent is not treated as a utility maximizing actor) can be added to the outer loop.

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Annex A
Model symbols

Here, we summarize the symbols of variables, functional forms and coefficients of the model.

Indices:

i	actors (1,…,m)
k	commodities (1,…,r)
n	random sources (1,…,N)
t	periods (1,…,T)

Variables:

d_t	m-vector of investment levels in period t
d_{it}	investment level of actor i in period t (scalar)
$\bar{d}_{iT}$	exogenous investment level of actor i in period T
h_{it}	income of actor i in period t
p_t	r-vector of commodity prices in period t
$\tilde{p}_t$	r-vector of price parameters in period t
$\bar{t}_{it}$	net exogenous transfer receipts of actor i in period t (scalar, expressed in baskets), with $\sum_i \bar{t}_{it} = 0$
x_t	m x r matrix of consumption in period t
x_{it}	r-vector of consumption of actor i in period t
V_{it}	budget deficit of actor i in period t
Z_{it}	upper bound on borrowing (t=1) or insurance (t=2) for actor i in period t
α	m-vector of welfare weights
α_i	welfare weight of actor i (scalar)
λ_i	shadow price of intertemporal budget constraint of actor i (scalar)
μ_{it}	shadow price (in terms of utility) of liquidity constraint of actor i in period t (t < T)
$\breve{\mu}_t$	m-vector of shadow prices (in terms of welfare) of liquidity constraints in period t (t < T)
$\breve{\mu}_{it}$	shadow price (in terms of welfare) of liquidity constraint of actor i in period t (t < T)
ω_{it}	r-vector of endowments of actor i in period t (t > 1)
$\bar{\omega}_{i1}$	r-vector of exogenous endowments of actor i in period 1
$\bar{\omega}_{it}^{\circ}$	r-vector of exogenous non-stochastic endowments of actor i in period t (t > 1)

Functional forms:

f	density function of the random event of period 2
f_i	actor i's subjective density function of the random event of period 2
f_i^*	ratio of subjective density function f_i to density function f

f^n	marginal density function of random source n of period 2
f_i^n	actor i's subjective marginal density function of random source n of period 2
g_i	vector-function (dimension r) of investment-induced supply expansion of actor i
u_{it}	instantaneous utility function of actor i in period t
v	iso-elastic transformation function
w_i	Stone-Geary function of actor i
$\overline{\omega}_{it}$	vector-function (dimension r) of exogenous endowments of actor i in period t (t > 1)

Coefficients:

ℓ_{it}	coefficient in liquidity constraint of actor i in period t
β_{ik}	budget share of commodity k in uncommitted consumption of actor i, within each period
δ	time preference parameter
γ_{ik}	committed level of consumption of commodity k by actor i
η_i	r-vector of commodity demand per unit of investment of actor i
ν	elasticity of supply expansion with respect to investment
ϑ	r-vector with composition of commodity basket (to express transfers)
σ	parameter of the iso-elastic utility transformation
τ_n	parameter of uniform density function of random source n
ψ_i	degree of optimism or pessimism in subjective density function
ζ_i	reference rate (at $d_{i1} = 1$) of investment-induced supply expansion of actor i

Other symbols:

I_1	information available in period 1
ε_2	random event (N-vector of random sources) in period 2
$\iota(k)$	pointer of sector k to one of the random sources
s	very small positive value

Annex B
First-order conditions of the utility programme

We rewrite problem (2.1) as follows:

$$\max_{x_{i1},x_{i2}(\varepsilon_2),d_{i1}\geq 0;\, V_{i1},V_{i2}(\varepsilon_2)} \quad u_{i1}(x_{i1})+\int u_{i2}(x_{i2}(\varepsilon_2))f_i(\varepsilon_2\,|I_1)d\varepsilon_2$$

s.t. $Z_{i1}-V_{i1}\geq 0$

$$Z_{i2}-V_{i1}-V_{i2}(\varepsilon_2)\geq 0$$

$$-V_{i1}-\int V_{i2}(\varepsilon_2)f(\varepsilon_2)d\varepsilon_2\geq 0$$

$$V_{i1}-p_1(x_{i1}+\eta_i d_{i1})+h_{i1}=0$$

$$V_{i2}(\varepsilon_2)-p_2(\varepsilon_2)\,[x_{i2}(\varepsilon_2)+\eta_i\bar{d}_{i2}-\overline{\omega}_{i2}(\varepsilon_2)-g_i(d_{i1},\varepsilon_2)-\bar{t}_{i2}\vartheta]=0$$

in which variables $h_{i1},p_1,p_2(\varepsilon_2),\bar{d}_{i2}$ are exogenous and positive, variables $Z_{i1},Z_{i2},\overline{\omega}_{i2}(\varepsilon_2)$ exogenous and non-negative, variable $\bar{t}_{i2}$ exogenous without sign constraint and parameters η_i and ϑ non-negative. We assume that u_{i1},u_{i2} and g_i are strictly concave and continuously differentiable in the decision variables , that density functions f and f_i are defined on a finite set (compact support) and that u_{i1} and u_{i2} are such that in the optimal solution each commodity is consumed by at least one of the actors, in each period.

If ε_2 would assume only a finite number of discrete values, first-order conditions could be derived with the standard Kuhn-Tucker theorem of nonlinear programming. However, the density functions may be defined over a continuous set implying that the number of arguments of (2.1) is not finite. Therefore, we have to resort to a more general class of optimization theorems. Here, we apply a theorem from the field of optimal control, as described in Takayama (1974).

The basic theorem of optimal control, Pontryagin's maximum principle,[28] cannot be applied directly to our problem since it does not allow for integrals in constraints or for variables that do not depend on ε_2. Therefore, we apply an extended version, due to Hestenes and included as theorem 8.C.4. in Takayama's book. In terms of this theorem, x_{i1},d_{i1} and V_{i1} are control

[28] Takayama (1974) has two alternative formulations of the maximum principle, i.e. theorems 8.A.1 and 8.A.6.

parameters and $x_{i2}(\varepsilon_2)$ and $V_{i2}(\varepsilon_2)$ are control variables, whereas the problem has no state variables i.e. no variables with a differential equation and terminal conditions.[29]

Density functions f and f_i have compact support. Hence, the integrals are taken over a finite range of ε_2 with exogenous bounds. We consider multipliers $\xi_i, \varphi_{i1}, \varphi_{i2}, \varphi_{i3}, \chi_{i1}(\varepsilon_2)$ and $\chi_{i2}(\varepsilon_2)$, and define, as in theorem 8.C.4, functions H, L and Ψ:

$$H\,[x_{i2}(\varepsilon_2), V_{i2}(\varepsilon_2), \varepsilon_2, \xi_i, \varphi_{i2}] =$$

$$\xi_i\, u_{i2}(x_{i2}(\varepsilon_2)) f_i(\varepsilon_2\,|I_1) - \varphi_{i2} V_{i2}(\varepsilon_2)\, f(\varepsilon_2)$$

$$L\,[x_{i2}(\varepsilon_2), V_{i2}(\varepsilon_2), d_{i1}, V_{i1}, \varepsilon_2, \xi_i, \varphi_{i2}, \chi_{i1}(\varepsilon_2), \chi_{i2}(\varepsilon_2)] =$$

$$H\,[x_{i2}(\varepsilon_2), V_{i2}(\varepsilon_2), \varepsilon_2, \xi_i, \varphi_{i2}] \quad +$$

$$\chi_{i1}(\varepsilon_2)\,[Z_{i2} - V_{i1} - V_{i2}(\varepsilon_2)] \quad +$$

$$\chi_{i2}(\varepsilon_2)\{\, V_{i2}(\varepsilon_2) - p_2(\varepsilon_2)\,[x_{i2}(\varepsilon_2) + \eta_i \bar{d}_{i2} - \bar{\omega}_{i2}(\varepsilon_2) - g_i(d_{i1}, \varepsilon_2) - \bar{t}_{i2}\vartheta]\}$$

$$\Psi(x_{i1}, d_{i1}, V_{i1}, \xi_i, \varphi_{i1}, \varphi_{i2}, \varphi_{i3}) =$$

$$\xi_i u_{i1}(x_{i1}) + \varphi_{i1}(Z_{i1} - V_{i1}) - \varphi_{i2} V_{i1} + \varphi_{i3}[V_{i1} - p_1(x_{i1} + \eta_i d_{i1}) + h_{i1}]$$

The function H is the Hamiltonian (reduced compared to a traditional optimal control problem since state variables are absent) which collects the integrands of the problem. The function L extends the Hamiltonian and is suitable as Lagrangian when optimal values of the control variables in a given event must be determined. The function Ψ is formulated in terms of the control parameters, with the same shadow prices as the Hamiltonian.

The differentiability and rank conditions of the theorem are satisfied, due to the assumptions stated earlier.[30] Therefore, in an optimal solution of control parameters and control variables, multipliers $\xi_i, \varphi_{i1}, \varphi_{i2}, \varphi_{i3}, \chi_{i1}(\varepsilon_2)$ and $\chi_{i2}(\varepsilon_2)$ should exist such that:

i) $\varphi_{i1} \geq 0 \quad \perp \quad V_{i1} \leq Z_{i1}$ (b.1)

[29] In this respect, optimal control theory is too broad for what we actually need. From an other perspective, it is not broad enough since it does not explicitly address multiple integrals.

[30] Assuming that our specification with piecewise continuity of the subjective density function is not harmful to the theorem.

ii) $\varphi_{i2} \geq 0 \quad \perp \quad V_{i1} + \int V_{i2}(\varepsilon_2)\, f(\varepsilon_2)\, d\varepsilon_2 \leq 0$ (b.2)

iii) $V_{i1} - p_1(x_{i1} + \eta_i d_{i1}) + h_{i1} = 0$ (b.3)

(iv) for each ε_2 the control variables $x_{i2}(\varepsilon_2)$ and $V_{i2}(\varepsilon_2)$ are optimal in

$$\max_{\hat{x}_{i2}(\varepsilon_2)\geq 0, \hat{V}_{i2}(\varepsilon_2)} H[\hat{x}_{i2}(\varepsilon_2), \hat{V}_{i2}(\varepsilon_2), \varepsilon_2, \xi_i, \varphi_{i2}]$$

$$\text{s.t.} \quad \hat{V}_{i2}(\varepsilon_2) - p_2(\varepsilon_2)\,[\hat{x}_{i2}(\varepsilon_2) + \eta_i \bar{d}_{i2} - \bar{\varpi}_{i2}(\varepsilon_2) - g_i(d_{i1}, \varepsilon_2) - \bar{t}_{i2}\vartheta] = 0$$

$$Z_{i2} - V_{i1} - \hat{V}_{i2}(\varepsilon_2) \geq 0$$

v) $-\dfrac{\partial \Psi}{\partial x_{i1}} \geq \int \dfrac{\partial L}{\partial x_{i1}} d\varepsilon_2 \quad \perp \quad x_{i1} \geq 0$ i.e.

$$-\xi_i \frac{\partial u_{i1}}{\partial x_{i1}} + \varphi_{i3}\, p_1 \geq 0 \quad \perp \quad x_{i1} \geq 0 \qquad \text{(b.4)}$$

vi) $-\dfrac{\partial \Psi}{\partial d_{i1}} \geq \int \dfrac{\partial L}{\partial d_{i1}} d\varepsilon_2 \quad \perp \quad d_{i1} \geq 0$ i.e.

$$\varphi_{i3}\, p_1 \eta_i \geq \int \chi_{i2}(\varepsilon_2)\, p_2(\varepsilon_2) \frac{\partial g_i(d_{i1}, \varepsilon_2)}{\partial d_{i1}} d\varepsilon_2 \quad \perp \quad d_{i1} \geq 0 \qquad \text{(b.5)}$$

vii) $-\dfrac{\partial \Psi}{\partial V_{i1}} = \int \dfrac{\partial L}{\partial V_{i1}} d\varepsilon_2$ i.e. $\varphi_{i1} + \varphi_{i2} - \varphi_{i3} = -\int \chi_{i1}(\varepsilon_2)\, d\varepsilon_2$ (b.6)

Conditions (v) – (vii) are the ‘transversality conditions’ of theorem 8.C.4. In our case, the interpretation of these conditions is different than in a traditional optimal control problem, mainly due to the absence of state variables and the different interpretation of time. In fact, (b.4) – (b.6) describe the intertemporal equality of costs and revenues. Condition (iv) can be elaborated upon in terms of the function L, leading to the traditional Kuhn-Tucker conditions:

$$\frac{\partial L}{\partial x_{i2}(\varepsilon_2)} \leq 0 \quad \perp \quad x_{i2}(\varepsilon_2) \geq 0 \qquad \text{i.e.}$$

$$\xi_i \frac{\partial u_{i2}}{\partial x_{i2}(\varepsilon_2)} f_i(\varepsilon_2 \,|I_1) - \chi_{i2}(\varepsilon_2)\, p_2(\varepsilon_2) \leq 0 \quad \perp \quad x_{i2}(\varepsilon_2) \geq 0 \qquad \text{(b.7)}$$

$$\frac{\partial L}{\partial V_{i2}(\varepsilon_2)} = 0 \qquad \text{i.e.} \; -\phi_{i2} f(\varepsilon_2) - \chi_{i1}(\varepsilon_2) + \chi_{i2}(\varepsilon_2) = 0 \tag{b.8}$$

$$\chi_{i1}(\varepsilon_2) \geq 0 \quad \perp \quad V_{i1} + V_{i2}(\varepsilon_2) \leq Z_{i2} \tag{b.9}$$

$$V_{i2}(\varepsilon_2) - p_2(\varepsilon_2)\,[x_{i2}(\varepsilon_2) + \eta_i \bar{d}_{i2} - \overline{\omega}_{i2}(\varepsilon_2) - g_i(d_{i1}, \varepsilon_2) - \bar{t}_{i2}\vartheta] = 0 \tag{b.10}$$

The factor ξ_i is merely a scaling factor of the utility functions. Therefore, it is not identifiable in the set of equations and we may assume any positive value. For convenience, we take the value 1. Conditions (b.1) – (b.10) are equivalent to conditions (2.4a) – (2.4f) plus the definitions of V_{i1} and $V_{i2}(\varepsilon_2)$. The equivalence can be checked by setting

$$\lambda_i = \varphi_{i2}, \;\; \mu_{i1} = \varphi_{i1} \text{ and } \mu_{i2}(\varepsilon_2) = \chi_{i1}(\varepsilon_2)/f(\varepsilon_2).$$

Takayama claims, for all the variants of the maximum principle discussed by him, the existence of a simple but powerful sufficiency theorem which guarantees optimality of the first-order conditions when the relevant functions are concave.[31] Given our assumptions, this indeed implies that conditions (b.1) – (b.10) are sufficient for a local optimum of utility programme (2.1).

[31] This claim is made in the introduction of section 8.C.c. Takayama elaborates the sufficiency theorem for only one specific case which has no control parameters and no equality constraints.

Annex C
Envelop theorem in a programme with feedback

Let $x_1,...,x_m,\lambda,\tilde{\lambda} \in R^r$, $a \in R$ and $\mu \in R^m$, whereas $f, g_1,..,g_r$ are real-valued functions on R^{mr+1}, $z_1,..,z_m$ real-valued functions on R and $y_1,...y_m$ vector-valued functions from R to R^r. We consider the following maximization programme, with shadow prices between brackets:

$$\max_{x_1,..,x_m} f(x_1,...,x_m,a)$$

s.t. $g_k(x_1,...,x_m,a) \geq 0$ $\quad(\lambda_k)\quad$ k=1,…,r

$\tilde{\lambda}x_i \leq \tilde{\lambda}y_i(a) + z_i(a)$ $\quad(\mu_i)\quad$ i=1,…,m

with feedback rule $\tilde{\lambda} = \lambda$.

We assume that all functions are continuously differentiable, that the Kuhn-Tucker conditions are necessary and sufficient for a maximum and that the optimal decision variables and shadow prices are unique. The optimal variables and shadow prices are functions of the parameter a, denoted as $\hat{x}_1(a),...,\hat{x}_m(a)$ and $\hat{\lambda}(a),\hat{\mu}(a)$ respectively. We consider values of a for which $\hat{\lambda}(a)$ is differentiable. Below, we will obtain the derivative of the optimum $M(a)$ with respect to a. The result generalizes the envelop theorem as given in e.g. Takayama (1974), Varian (1984) or Ginsburgh and Keyzer (1997), in the sense that it adds a feedback rule to the programme.

By definition:

$$\text{(i)}\quad \frac{dM(a)}{da} = \Sigma_i \frac{\partial f(\hat{x}_1(a),...,\hat{x}_m(a),a)}{\partial x_i}\frac{d\hat{x}_i(a)}{da} + \frac{\partial f(\hat{x}_1(a),...,\hat{x}_m(a),a)}{\partial a}$$

Kuhn-Tucker conditions:

$$\text{(ii)}\quad \frac{\partial f(\hat{x}_1(a),...,\hat{x}_m(a),a)}{\partial x_i} + \Sigma_k \hat{\lambda}_k(a)\frac{\partial g_k(\hat{x}_1(a),...,\hat{x}_m(a),a)}{\partial x_i} - \hat{\mu}_i(a)\tilde{\lambda}(a) = 0 \qquad i=1,..,m$$

$$\text{(iii)}\quad g_k(\hat{x}_1(a),...,\hat{x}_m(a),a) \geq 0 \quad \perp \quad \hat{\lambda}_k(a) \geq 0 \qquad k=1,\ldots,r$$

$$\text{(iv)}\quad \tilde{\lambda}(a)\hat{x}_i(a) \leq \tilde{\lambda}(a)y_i(a) + z_i(a) \quad \perp \quad \hat{\mu}_i(a) \geq 0 \qquad i=1,\ldots,m$$

Furthermore:

(v) $\tilde{\lambda}(a) = \hat{\lambda}(a)$

Since (iii) has to hold for minor variations in parameter a, we have for k=1,..,r

(vi) $\hat{\lambda}_k(a) = 0$ or $\sum_i \frac{\partial g_k(\hat{x}_1(a),...,\hat{x}_m(a),a)}{\partial x_i}\frac{d\hat{x}_i(a)}{da} + \frac{\partial g_k(\hat{x}_1(a),...,\hat{x}_m(a),a)}{\partial a} = 0$

For the same reason we have from (iv), for i=1,…,m

(vii) $\hat{\mu}_i(a) = 0$ or $\tilde{\lambda}(a)\frac{d\hat{x}_i(a)}{da} + \hat{x}_i(a)\frac{d\tilde{\lambda}(a)}{da} = \tilde{\lambda}(a)\frac{dy_i(a)}{da} + y_i(a)\frac{d\tilde{\lambda}(a)}{da} + \frac{dz_i(a)}{da}$

Also the feedback relation has to hold for minor variations in a, hence

(viii) $\frac{d\tilde{\lambda}(a)}{da} = \frac{d\hat{\lambda}(a)}{da}$

Substitution of (ii) in (i) gives:

$$\frac{dM(a)}{da} = \frac{\partial f(\hat{x}_1(a),...,\hat{x}_m(a),a)}{\partial a} + \sum_i\left[-\sum_k \hat{\lambda}_k(a)\frac{\partial g_k(\hat{x}_1(a),...,\hat{x}_m(a),a)}{\partial x_i} + \hat{\mu}_i(a)\tilde{\lambda}(a)\right]\frac{d\hat{x}_i(a)}{da}$$

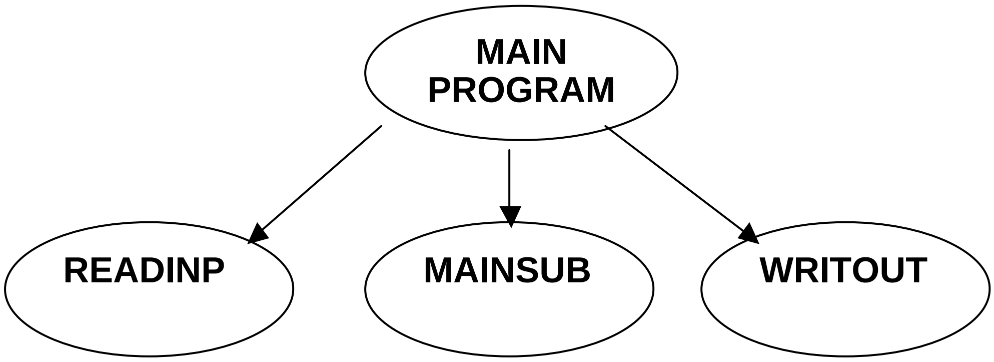
Further substitution, using (vi) – (viii):

$$\frac{dM(a)}{da} = \frac{\partial f(\hat{x}_1(a),...,\hat{x}_m(a),a)}{\partial a} + \sum_k \hat{\lambda}_k(a)\frac{\partial g_k(\hat{x}_1(a),...,\hat{x}_m(a),a)}{\partial a} +$$

$$\sum_i \hat{\mu}_i(a)\left[\hat{\lambda}(a)\frac{dy_i(a)}{da} + \frac{dz_i(a)}{da} + \frac{d\hat{\lambda}(a)}{da}(y_i(a) - \hat{x}_i(a))\right]$$

The terms with $d\hat{\lambda}(a)/da$ make up the difference with the envelop theorem for a program without feedback. Only if $d\hat{\lambda}(a)/da = 0$ or $\hat{\mu}(a) = 0$, there is no effective difference.

Annex D
Overview of Fortran programme



Data input   Model solution   Report writing

MAINSUB (model solution) :

- specify model scenario
- initialize welfare weights (and other variables outer loop)
- (*) solve model at given welfare weights → SOLFIXW, mode 1
- adjust welfare weights (and other variables outer loop)
- check convergence of outer loop: if NO → back to (*)

  if YES → SOLFIXW, mode 2

SOLFIXW, mode 1 (model solution at given welfare weights) :

- initialization period-1 variables
- (*) perform random drawings → RDRAW
- solve period 2 at given random event and given period-1 variables → FFSOL2 or ANSOL2
- update period-2 expectations
- update period-1 variables
- check convergence of middle loop: if NO → back to (*)

  if YES → back to MAINSUB

SOLFIXW, mode 2 (ex-post calculations) :

- price normalization of period-1 solution
- Monte-Carlo simulation of period 2 (period-1 given):
  - (*) perform random drawings → RDRAW
  - solve period 2 at given drawing → FFSOL2 or ANSOL2
  - update period-2 expectations
  - check iteration count: if not yet final → back to (*)
    if final → back to MAINSUB

FFSOL2 (solve period 2 at given welfare weights, period-1 variables and random drawing, in case that $\sigma \neq 1$) :

- initialize period-2 prices and liquidity shadow prices
- (*) calculate period-2 consumption
- adjust period-2 prices and liquidity shadow prices
- check convergence: if NO → back to (*)
  if YES → back to SOLFIXW

ANSOL2 (solve period 2 at given welfare weights, period-1 variables and random drawing, in case that $\sigma = 1$) :

- initialize liquidity shadow prices
- (*) calculate period-2 consumption and period-2 prices
- adjust liquidity shadow prices
- check convergence: if NO → back to (*)
  if YES → back to SOLFIXW

RDRAW (random event) :

- draw random sources (together the random event)
- calculate effect of random event on each sector

Annex E
Technical model report

Here, we present by way of example, the technical report of model run D of table 8.1.

```
MODEL SPECIFICATION AND CONVERGENCE REPORT:  BLIFRE2.150
 --------------------------------------------------------------------------------
 Model option:      Rational expectations

 Number of random sources:    8          Investment function not stochastic

 Parameters tau and correlation matrix:

             EVENT1    EVENT2    EVENT3    EVENT4    EVENT5    EVENT6    EVENT7    EVENT8
   Tau        .50       .50       .25       .25       .25       .25       .10       .10
   FOODAGRIC    1         0         0         0         0         0         0         0
   CASHAGRIC    0         1         0         0         0         0         0         0
   MINING       0         0         1         0         0         0         0         0
   FUEL         0         0         0         1         0         0         0         0
   LOCMANUF     0         0         0         0         1         0         0         0
   IMPMANUF     0         0         0         0         0         1         0         0
   CONSTRUCT    0         0         0         0         0         0         1         0
   SERVICES     0         0         0         0         0         0         0         1

 Parameters:                Sigma    Liquidity-1  Liquidity-2
             RURAL          1.500         .050       25.000
             URBANLOW       1.500         .050       25.000
             URBANHIGH      1.500         .050       25.000
             GOVERNMENT     1.500         .100       25.000
 --------------------------------------------------------------------------------
 Results from welfare iteration  5000
 *** Warning: MAXITW reached

Expected welfare              10225.865
Investment levels D(I,1)
 INVEST  RURAL            2.284
 INVEST  URBANLOW         2.511
 INVEST  URBANHIGH         .573
 INVEST  GOVERNMENT        .784

Test on first-order conditions of period  one in welfare iteration  5000

   Equality of marginal consumption costs and returns:

                                 RURAL       URBANLOW    URBANHIGH   GOVERNMENT
      FOODAGRIC   Costs            .993         .993         .993         .993
      FOODAGRIC   Returns          .993         .993         .993         .000

      CASHAGRIC   Costs            .684         .684         .684         .684
      CASHAGRIC   Returns          .684         .684         .684         .000

      MINING      Costs            .571         .571         .571         .571
      MINING      Returns          .571         .571         .571         .571

      FUEL        Costs           1.825        1.825        1.825        1.825
      FUEL        Returns         1.825        1.825        1.825         .000

      LOCMANUF    Costs            .895         .895         .895         .895
      LOCMANUF    Returns          .895         .895         .895         .000
```

IMPMANUF	Costs	3.390	3.390	3.390	3.390
IMPMANUF	Returns	3.390	3.390	3.390	.000
CONSTRUCT	Costs	.129	.129	.129	.129
CONSTRUCT	Returns	.129	.129	.129	.000
SERVICES	Costs	1.141	1.141	1.141	1.141
SERVICES	Returns	1.141	1.141	1.141	1.141

Equality of marginal investment costs and returns:

RURAL	78.619	78.491
URBANLOW	24.239	24.221
URBANHIGH	69.698	69.801
GOVERNMENT	114.529	115.107

Complementarity between excess supply and prices:

FOODAGRIC	-.001	.993
CASHAGRIC	.000	.684
MINING	.000	.571
FUEL	.000	1.825
LOCMANUF	.000	.895
IMPMANUF	.000	3.390
CONSTRUCT	.000	.129
SERVICES	.000	1.141

Complementarity between excess liquidity and shadow prices:

RURAL	24.874	.000
URBANLOW	16.525	.000
URBANHIGH	95.857	.000
GOVERNMENT	43.493	.000

Summary welfare iteration 5000

Actor	Alpha	Budget gap	Relative gap	Totsupval	Inv 1
RURAL	163.1559	.5421	.0002	2992.0405	2.284
URBANLOW	85.9701	.2957	.0003	1081.5398	2.511
URBANHIGH	114.3047	-.1418	-.0001	1422.3910	.573
GOVERNMENT	42.9651	-.7333	-.0010	733.7461	.784

Actor	Liqbound 1	Last change	Liqbound 2	Last change
RURAL	76.1862	-.0001	38044.2080	-.8151
URBANLOW	27.8397	-.0001	13909.8904	-.4785
URBANHIGH	37.5529	-.0001	18836.5758	-.7166
GOVERNMENT	39.1728	-.0001	9799.5589	-.5967

Evaluation of random drawings:

Source	Drawings	Average	Variance
EVENT1	1340927	-.00035	.08327
EVENT2	1340927	-.00037	.08340
EVENT3	1340927	-.00024	.02082
EVENT4	1340927	-.00004	.02081
EVENT5	1340927	-.00016	.02085
EVENT6	1340927	-.00005	.02083
EVENT7	1340927	.00006	.00333
EVENT8	1340927	.00003	.00333

Algorithmic summary:

Loop	Calls	Iterations per call
WELFARE	1	5000
PERIOD1-SQG	5000	268
PERIOD2	1340927	79

The Centre for World Food Studies (Dutch acronym SOW-VU) is a research institute related to the Department of Economics and Econometrics of the Vrije Universiteit Amsterdam. It was established in 1977 and engages in quantitative analyses to support national and international policy formulation in the areas of food, agriculture and development cooperation.

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SOW-VU's research record consists of a series of staff working papers (for mainly internal use), research memoranda (refereed) and research reports (refereed, prepared through team work).

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