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Consistent calculation, valuation and calibration of surface flows in spatially explicit and dynamic models

by

M.A. Keyzer

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Contents

Abstract

The paper presents an algorithm to solve a spatially explicit dynamic model in which substances flow downward over a relief of arbitrary shape and material balances are maintained in every cell of the spatial grid. On the basis of the downstream extraction prices and amenity values, the algorithm, implemented in Fortran, also calculates in a numerically effective way the upstream prices of the flows, for problems of very large dimensions, say, with a few million cells. We distinguish between a linear version, in which the directional outflow fractions of every cell are kept independent of the flows, and a nonlinear one, which may also exhibit discontinuities, and can accommodate an arbitrary valuation criterion that is not necessarily separable over sites. Next, we indicate how, at given prices of the flows, we can in a fully decentralized calculation on a site-by-site basis, impute prices of stocks and generally of model parameters, in the static version of the model as well as in the steady state of a dynamic version, with carryover stocks. Moreover, we present conditions under which the algorithm can determine the optimal extraction at every site as well optimal routing between sites, and locate sites where uphill pumping would be profitable. We also indicate that the valuation criterion that is being maximized might be a likelihood function, in which case the model parameters become the key variables for valuation, with their prices pointing to directions of change that would improve the fit to observed data.

Section 1
Introduction[1]

Current attempts at building spatially explicit hydrology models face an inherent curse of dimensionality, as the number of grid cells needed to represent the relief and other measures of spatial diversity is necessarily large, especially when the data set is a satellite image. Moreover, hydrological models are dynamic by nature, and, must, therefore, distinguish multiple time periods. While traditional hydrological models focus on the dynamics of a limited number of geographical units, within a watershed, say, in the order of twenty, the subclass of spatially explicit models follow the details of the relief, which easily leads to grids of hundreds of thousands of cells. In these models, preservation of physical consistency is a central issue that can be dealt with relatively easily in models that are gravity driven, since this fundamentally maintains a recursive structure. Nonetheless, it poses major problems in an optimization context, since the number of water balances to be maintained amply exceeds the capacity of regular mathematical programming algorithms. Consequently, the optimal control of flows, the economic assessment of scarcities and the estimation of model parameters are faced with major difficulties.

The hydrological literature pays much attention to the adequate representation of flows between cells, usually as an approximation of partial differential equations (e.g. Holder et al., 2000). One might expect that when simulations are conducted on a fine grid, it becomes possible to keep the model of the individual grid cell relatively simple, since the spatial continuum is approximated sufficiently closely to calculate gradients and second-order derivatives from function differences. Yet it appears that in practice the grid is almost always too coarse for this, and dedicated computational techniques have to be invoked to obtain a realistic temporal and spatial outflow pattern. A further difficulty is that detailed data records are only available at specified points in the form of a delivery profile over time (hydrograph). Calibration would be much easier, if the full model could be brought under optimization. This would also permit to find optimal rules for water control and economic valuation and this requires solving very large problems.

Specifically, the issue is the following. Consider a spatially explicit water model, for a single time period, on an $S = G \times G$ grid, with contiguous cells indexed s, of a given shape.[2] For given, nonnegative, net precipitation b_s at site s – rainfall minus evaporation and flow independent extraction – the water flow q_s available for delivery to other cells is the sum of accruals y_s from other cells and this net precipitation:

$$q_s = y_s + b_s . \tag{1.1}$$

[1] The author thanks Ben Sonneveld for comments on an earlier draft.
[2] Hence, every variable with index s constitutes a geographical map.

For ease of notation, we take y_s to be a scalar but we will point out that the procedures to be presented readily extend to the case where it is a vector with multiple substances. We will also discuss dynamics in some detail, but already note here that the static models to be presented would also apply for locations indexed st, with S sites and T time periods, and each period situated downstream of its predecessor.

Every destination site d receives nonnegative fraction a_{ds} of q_s from source site $s = 1,\ldots,S$.

$$y_d = \sum_s a_{ds} q_s \ . \tag{1.2}$$

These fractions are possibly determined through an elaborate hydrological model that also determines the retention coefficient $r_s = 1 - \sum_d a_{ds}$, as well as by economic consideration but in the linear version of the model, the essential requirement is that they should not depend on q_s.[3] Because of inevitable spatial aggregation within cells and randomness over time it seems advisable to allow for α-shares that are positive for more than one destination, so as to allow the flow to spread rather than follow the steepest descent dictated by gravity theory. In fact, the fractions $a_{ds} /(1 - r_s)$ can, for $r_s < 1$, be interpreted as the probability of water flowing from site s to destination d.

We seek a solution to (1.1) and (1.2). This linear system can be written in matrix form as:

$$y = A(y + b)\ , \tag{1.3}$$

where A is the matrix of coefficients a_{ds}, y the vector of accruals, belonging to the compact set $Y \subset R_+^S$. Hence, it would seem straightforward to solve for y and q. However, as mentioned above the difficulty is that the size of G easily exceeds thousand, in which case the matrix A has $G^4 = 10^{12}$ cells, indicating that the problem should not be addressed by conventional matrix manipulations and cannot be supplied as constraint to a conventional optimization package.

In practice, hydrological models impose an a priori order of visits to the sites of the system of equations (1.1)-(1.2), taking the then available value of accruals as given (Wilson, 2001), either following the grid or following the contour lines in top down order (see e.g. Menduni et al. 2002). Specifically, the imposition of a fixed sequence of calculations whereby every location is only visited once, amounts to performing, in a given visiting order $d = 1,\ldots,S$, a single iteration k of the system of equations:

$$y_d^{k+1} = \sum_s a_{ds}(y_s^k + b_s)\ , \qquad k = 1, 2, \ldots, \tag{1.4}$$

[3] Flows bifurcate whenever the share is non-zero for more than one destination. This is more easily represented on a spatial grid than in a spatial continuum, hence the discrete treatment of space.

where k is a time period of, say, a day or a week. Now, if water flows between two cells are downhill only, the relationships between all cells become recursive. This implies that matrix A can be rearranged by decreasing altitude into lower triangular form, with the first row and column pointing to the site with highest elevation, and a single iteration is sufficient to solve (1.4), because the right hand side y_s^k is known whenever a_{ds} is positive.[4] The challenge is to exploit this property in valuation and optimization algorithms.

Note that this rearrangement means imposition of the sequence of visits that follows the contour lines from top to bottom. Any other sequence will lead to inconsistency after a single iteration. Yet, the fact that some water always leaves the system implies that the matrix A, and more generally the Jacobian of the accrual function is non-negative and has dominant eigenvalue less than unity (see e.g. Ortega and Rheinboldt, 1970, ch. 8,12). This ensures that, also if the row and columns are not rearranged or if water flows in both directions between two cells, the iteration converges to a unique (equilibrium) solution, that approximately solves (1.3) over a longer period.

Building on Hoekstra et al. (2000) and Keyzer (2000), the present paper formulates an algorithmic approach for consistent calculation, valuation and calibration of gravity driven flows. It draws on concepts from classical valuation theory, similar to those used in early labor theories of value (Sraffa, 1960; Morishima, 1973), as well as from modern capital theory (Stokey and Lucas, 1989), to exploit the recursive nature of the problem.

Overview

Section 2 starts with a formalization of the sequential approach in (1.4), that exactly solves (1.3) in a finite number of iterations, possibly for multiple substances and that flow downward over a relief of arbitrary shape. Hence, we limit attention to the case with unilateral flows between cells. Section 3 applies duality theory to calculate the prices of the flows, following the contour lines, but in bottom to top. Section 4 considers a nonlinear version, in which the directional outflow fractions of every cell are dependent on the flows, may exhibit discontinuities, and can accommodate an arbitrary valuation criterion that is not necessarily separable over sites. Section 5 indicates how, at given prices of the flows, we can in a fully decentralized calculation on a site-by-site basis, impute prices of stocks and generally of model parameters, in the static version of the model, and for the case with random shocks. Section 6 extends the algorithm to deal with the steady state of a dynamic version, specified over an infinite time horizon, with carryover stocks. The critical step is to calculate the price of these stocks. Section 7 presents conditions under which the algorithm can determine the optimal extraction at every site as well as optimal routing between sites, and locate sites where uphill pumping would be profitable. We also indicate that the valuation criterion that is being maximized might be a likelihood function, in which case the

[4] If the matrix cannot be reordered there is a circularity in the flows. In this case, it still is possible to solve recursively by applying Gaussian elimination to the blocks of cells with circularity.

model parameters become the key variables for valuation, with their prices pointing to directions of change that would improve the fit to observed data. So far, the various algorithms were implemented in Fortran for the linear version of the model. Section 8 concludes.

Section 2
Model solution

We consider a two-dimensional grid that divides the region $\overline{B}$ into non-overlapping cells with territory B_s: $\bigcup_{s=1}^{S} B_s = \overline{B} \subset R^2$, and altitude a_s. The region is taken to consist of a number of complete watersheds. This makes it possible to disregard inflows and outflows to and from other regions. The set of neighbors of cell *s* is denoted by $N_s = \{ n \in [1,2,...,S] \mid B_n \cap B_s \neq \emptyset, n \neq s \}$, and $N_s^d = \{ n \in N_s \mid a_n < a_s \}$ and $N_s^u = \{ n \in N_s \mid a_n \geq a_s \}$ denote the subsets of downstream and upstream neighbors respectively. We will use the index n for neighbors, *d* for downstream neighbors and *u* for upstream neighbors. We suppose that these sets are known. If the cells that build up the grid consist of polygons, these will have to be supplied from a GIS-data base, but if they are fixed rectangles, it is possible to use the known latitude, and the longitude to identify the eight neighbors as $N_s = \{ n \in [1,2,...,S] \mid \left| x_n^{la} - x_s^{la} \right| \leq 1 \ \ and \ \left| x_n^{lo} - x_s^{lo} \right| \leq 1, n \neq s \}$, by direct calculation (union jack), and similarly for triangles or hexagons.

Three natural properties make it possible to ensure that the system of equations can be solved in a recursive manner: water only flows downwards (this may not be the case for groundwater), flows should be nonnegative, and water cannot skip adjacent points.

Assumption A (downward flow): The flows obey $a_{rs} = 0$ whenever $r \in N_s^u$.

Since the model can only describe downhill flows, it cannot accommodate negative net precipitation. Even though the actual requirement is that $y_s + b_s \geq 0$, we impose:

Assumption B (net precipitation): The net precipitation is nonnegative at every site: $b_s \geq 0$.

Now the key step is to define (and to compute) an altitude based ordering s_ℓ, $\ell = 1,...,C$, such that $a_{s_{\ell+1}} \leq a_{s_\ell}$. Hence, this ordering follows the contour map and is indifferent as to the ranking of sites of equal altitude. This is because gravity driven flows will not circulate between them.

Proposition 1 (computation of accruals) If assumptions A and B hold, then, starting from $y_d = 0$, for $d = 1,...,C$, the sequential C-step cumulative calculation that updates the accrual according to

$$y_d := a_{ds}(y_s + b_s) + y_d, \qquad \text{for } s = s_\ell,\ d \in N_s^d, \text{ and } \ell = 1,...,C, \tag{2.1}$$

solves model (1.1)-(1.2).

Proof. The proof proceeds by direct induction. Note that, by Assumption A, at $\ell = 1$, the highest point, we have no inflow, and hence $y_{s_1} = 0$ is the correct value. Then, we can compute the inflow into adjacent points. Next, at $\ell = 2$ we have already accounted for all possible inflow (from $\ell = 1$, or if this is not an adjacent point, none at all), and so on. Hence, we need exactly C iterations of (2.1) to solve the model. ■

By Assumption A, the evaluation of equation (1.3) itself is easy because it only involves the neighborhood of s_ℓ. The fractions a might be functions rather than constants, as the argument only required them not to depend on downstream y-values. Now given the equilibrium solution y^*, we may calculate the retention at every site as:

$$r_s = r_s(y_s^* + b_s), \tag{2.2}$$

where the retention fraction $r_s = 1 - \sum_{d \in N_s^d} a_{ds}$ follows residually. If the flows follow steepest descent, and the retention fraction is given by natural circumstances, possibly the relief, the fractions a obey:

$$max_{a_{ds} \geq 0} [\sum_{d \in N_s^d} (a_s - a_d) a_{ds} \mid \sum_{d \in N_s^d} a_{ds} + r_s = 1] \tag{2.3}$$

Note that this form cannot allocate water between two steepest descent destinations with the same altitude. Alternatively, if flows are distributed in proportion to the difference in altitude, they can be determined uniquely from:

$$max_{a_{ds} \geq 0} [\sum_{d \in N_s^d} (a_s - a_d) ln\, a_{ds} \mid \sum_{d \in N_s^d} a_{ds} + r_s = 1] \tag{2.4}$$

It is easy to apply any concave objective function, with coefficients reflecting the distribution of rainfall, and drainage flows within the different parts of the cell. It is also possible to apply some transformation of the difference in altitudes to reflect that steep slopes more closely obey steepest descent than flat ones. In applications, this function will have to be based on engineering information, and depend on coefficients that may need further calibration.

Note that the same sequence of calculations, and the same matrix notation as in (1.3), applies if the accruals form a vector, say, of different physical substances. This makes it possible to describe how these substances mix, dilute or deposit along the relief and how, say, the use by households or industry adds pollutants more than it extracts. It also permits to incorporate indicator variables, to compute, say, the average distance traveled by a drop of water, or the share by origin at a given destination. The model can also be used to locate sinks, for example to detect errors in a hydrological map. For this, one may consider an altitude map with a hydrological flow overlay. Now if one deducts, say, one meter for every location where water flows, any sink in the flow that is not a lake or a hole points to an error of the map.

Clearly, the recursive structure with fixed shares cannot accommodate counter-effects such as soil moisture saturation, whose representation calls for the introduction of nonlinearity (intra-period saturation) and stock variables (inter-period saturation) to be dealt with in sections 4 and 6, respectively.

Section 3
Valuation of flows

Suppose that every unit of water retained has extraction value p_s and the water used without retention has amenity value f_s. These unit values might be negative, to reflect, say, the damaging effects of floods or water pollution. The question is now to determine the use value of water at the different sites for given net precipitations. As described in Keyzer (2000), we can derive this value as the Lagrange multipliers p of the mathematical program:

$$\begin{aligned} & max_y (f^T + p^T R) y \\ & subject\ to \\ & y = A(y + b), \qquad\qquad (p\ ) \end{aligned} \tag{3.1}$$

where the superscript T denotes the transpose, and the matrix A is taken to be constant, while R denotes the diagonal matrix of retention factors. As before, the difficulty is that this program is far too large to be solved by conventional methods. Yet, to find the price we only need to solve the first-order (or dual) equations associated with this problem, which read:

$$p = A^T p + f + Rp, \tag{3.2}$$

indicating that the price of water is equal to the delivery value to other sites plus the amenity value plus the extraction value.

For computations, the sequence of calculations in (2.1) is reversed, and moves from bottom to top.

Proposition 2 (valuation of accruals) If assumptions A and B hold, then, starting from $p_s = 0$, for $s = 1,...,C$, the sequential C-step calculation:

$$p_s = \sum_{d \in N_s^d} a_{ds} p_d + f_s + r_s p_s, \qquad \text{for } s = s_\ell \text{, and } \ell = C, C-1,...,1 \tag{3.3}$$

solves equations (3.2).

Proof. The proof proceeds as for proposition 1, by induction. We start from the end-delivery at the lowest site, which necessarily is an end-sink .There, the price p_s is equal to amenity plus extraction value, since there is no lower site to be delivered to. Then, we turn to the next lowest site. It may or may not deliver to the earlier one. The value of one unit of water is equal to the sum over the fractions delivered to lower levels, multiplied by the prices at these levels, that are already available, plus the amenity and the extraction values, and so on. ■

The recursion does not imply that all prices are necessarily higher upstream. There are three counteracting forces at play. The first is dilution. Generally, water flows out without any use at some of the lower end sites, and, consequently, has zero amenity and extraction value. Upstream accrual prices p_s will reflect this. The second is retention, as expressed by $\sum_{d\in N_s^d} a_{ds} < 1$, which causes part of the downstream value to leak out. Finally, some accrual prices might be negative, because of the damages caused, say, by floods, that can lead to negative amenity and extraction prices.

It appears that a problem with one million cells could be solved both primally, by (1.3), and dually, by (3.1), within one minute on a PC. Once a solution has been obtained, it is possible to present value accounts by site, through multiplication of prices and quantities. Specifically, since (3.1) is a linear program, the value of its objective equals that of the its right-hand side resource *Ab*, the total value of net precipitation delivered (primal-dual equality):

$$(f^T + p^T R)y = p^T Ab\,, \tag{3.4}$$

and then, by (3.2), the value of the net precipitation b itself satisfies:

$$(f^T + p^T R)(y + b) = p^T b\,, \tag{3.5}$$

illustrating that the value of final use is fully attributed to the net precipitation, which in this linear framework is the only source of "wealth". Now we can attribute this value to the various sites. Some will have a deficit, others a surplus, which can be expressed as a transfer:

$$(f_s + p_s r_s)(y_s + b_s) = p_s b_s + q_s\,, \tag{3.6}$$

indicating that at every site the value in end-use (direct amenity plus extraction value) is equal to the total value $p_s b_s$ plus a, possibly negative, transfer received q_s, a balancing item that sums to zero over all cells. Hence, the *Value Account* or *Budget* by site can be written in tabular form as:

	Value in end use:	$(f_s + p_s r_s)(y_s + b_s)$
=		
	Value of net precipitation:	$p_s b_s$
+	Transfer received:	q_s

If every cell was full owner of its water, and had to trade it with its neigbors, it would earn q_s, i.e. pay the negative of this amount. We note for later reference that at prices p_s, every accrual y_s is "optimal" in the site-specific profit maximization problem, measured as the value of end-use plus deliveries, minus the cost of accruals:

$$P_s = max_{y_s}(f_s + r_s p_s)y_s + (\sum_{d\in N_s^d} a_{ds} p_d)(y_s + b_s) - p_s y_s \ , \tag{3.7}$$

which, by (3.2), reduces to a fixed "profit" $P_s = \sum_{d\in N_s^d} p_d a_{ds} b_s$.[5] This property permits to construct value accounts of accruals that show how the value that flows along site s, in contrast to the value account (3.6) that shows how much value is spent by the site and originates from it. The property will prove of particular relevance in the non-linear case.

Finally, we mention that in the general case where water flows in two directions between grid cells, it is, like in (1.4), possible to solve for the prices by the iteration:

$$p^{k+1} = A^T p^k + f + Rp, \qquad k = 1, 2, \ldots, \tag{3.8}$$

albeit that this obviously is more demanding computationally.

[5] Since the profit on delivery is zero, any y_s-value would be optimal. Yet the program can be used for accounting purposes.

Section 4
Model solution and valuation of flows in the nonlinear case

The valuation in (3.1) took the matrix A, to be fixed. Hydrological models are seldom linear and in the general situation we are faced with a model of the form:

$$y = H(y)(y+b), \tag{4.1}$$

where the hydrology function $H(y)$ is a matrix function. In this paper, we abstract from all the difficulties in obtaining such a function, and focus on the properties that are relevant for our purposes. To characterize it, we must refer to associated matrix. We will say that matrix B conforms to matrix A, if it has the same dimensions, and no non-zero entries at places where A has zero entries. When mentioning the matrix A, we refer to the matrix in equation (1.3), satisfying Assumption A. We can now make the required assumption:

Assumption F1: The matrix of the model function $H(y)$ conforms to A and satisfies assumption A, and every coefficient only depends on accruals at higher altitudes, i.e. is determined as $a_{ds_\ell} = h_{ds_\ell}(y_{s_1},...,y_{s_\ell}),\ \ell = 1,...,S$.

Under these conditions, the solution sequence is exactly as in Proposition 1.

Proposition 3 (computation of accruals) If Assumption F1 holds, then, starting from $y_d = 0$, for $d = 1,...,C$, the sequential C-step calculation, that updates the accruals according to

$$y_d := h_{ds}(y_{s_1},...,y_{s_\ell})(y_s + b_s) + y_d \text{, for } s = s_\ell,\ d \in N_s^d \text{, and } \ell = 1,...,C, \tag{4.2}$$

solves model (4.1).

Proof. The proof proceeds as for Proposition 1. ■

Turning to valuation, the nonlinear form of the associated program (3.1) is:

$$\begin{aligned} &\max_y \ w(y) \\ &\text{subject to} \\ &\quad y = H(y)(y+b), \qquad (p) \end{aligned} \tag{4.3}$$

where the objective (welfare) function $w(y)$ is supposed to satisfy:

Assumption W: The valuation objective $w(y)$ is differentiable almost everywhere on Y.

The requirement that the objective is differentiable almost everywhere means that the set of points where it is not differentiable has measure zero on *Y*. In practical terms, it is important to allow for this because the hydrology model may distinguish various regimes, and possibly contain discrete shifts in the direction of a flow. Conversely, it also emphasizes that the valuation method only reveals local properties of the model. Assumption W is also weak in another sense, since it does not impose any concavity restriction and allows the derivative of the objective to depend on the accrual at all sites simultaneously.

The marginal valuation can now be obtained from the first-order Kuhn-Tucker conditions w.r.t. *y* as:

$$w'^T = p^T(I - F'), \tag{4.4}$$

where

$$F(y) = H(y)(y + b), \tag{4.5}$$

and, therefore,

$$p = F'^T p + w'. \tag{4.6}$$

The flow function *F* satisfies

$$F_d(y) = \Sigma_s f_{ds}(y,b) = \Sigma_s h_{ds}(y)(y_s + b_s), \tag{4.7}$$

where h_{ds} is an element of the matrix *H*, and its derivative is

$$\frac{\partial F_d}{\partial y_r} = \Sigma_s \frac{\partial h_{ds}(y)}{\partial y_r}(y_s + b_s) + h_{dr}. \tag{4.8}$$

We have seen in the previous section, that the valuation requires the derivative F' to conform to *A*.[6] This shows that to maintain the recursive valuation of Proposition 2, the derivative should only contain elements on the diagonal:

Assumption F2: The hydrology function *H(y)* is differentiable almost everywhere on *Y*, and satisfies $\frac{\partial h_{ds}}{\partial y_r} = 0$, for $s \neq r$, whenever it is differentiable.

Hence, we may write $f_{ds}(y_s, b_s)$ and the *A*-matrix in (3.1) would have entries:

[6] In case the matrix does not conform to *A*, the iteration (1.4) could be used here as well, and will converge if the dominant eigenvalue of the Jacobian (4.7) is less than unity.

$$\tilde{a}_{ds} = \frac{\partial F_d}{\partial y_s} = \frac{\partial h_{ds}(y_s)}{\partial y_s}(y_s + b_s) + h_{ds}. \tag{4.9}$$

In the linear structure, upstream conditions do not affect the prices of downstream accruals, but as soon as either the objective $w(y)$ or the share matrix A become nonlinear, changes in upstream flows have an impact, either directly via second-order cross-effects that change off-diagonal elements in the Hessian matrix of w, and indirectly via the changes in downstream flows that induce changes in shares.[7]

Proposition 4 (valuation of accruals) If assumptions F1, F2, W hold, then, after running procedure (4.2), and starting from $p_s = 0$, for $s = 1,...,C$, the sequential C-step calculation:

$$p_s = \sum_{d \in N_s^d} \tilde{a}_{ds} p_d + w_s', \qquad \text{for } s = s_\ell \text{, and } \ell = C, C-1,...,1 \tag{4.10}$$

computes the Lagrange multipliers of program (4.3).

Proof. The proof proceeds as for proposition 2. ■

In addition, it is possible to attribute special costs and revenues to the bilateral flows between sites. The flow from s to d will take time, it might also involve other costs. The interest charge can be accounted for by means of a correction factor t_{ds} in the valuation, and the other cost by means of a deduction x_{ds} in (3.1)

$$\tilde{a}_{ds} = (1 - t_{ds}) a_{ds} - x_{ds}, \tag{4.11}$$

where the coefficients $\tilde{a}$ build the matrix A.

[7] In the general case where Assumption A does not hold, iteration (3.8) can be used to compute prices.

Section 5
Valuation of structural characteristics

We observe that we can as nonlinear equivalent of (3.7) formulate the site-specific program:

$$P_s(p, b_s) = max_{y_s}\ w'_s y_s + \sum_{d \in N_s^d} p_d f_{ds}(y_s, b_s) - p_s y_s\ , \tag{5.1}$$

where the profit is not necessarily zero. Now to ensure that the model solution is an optimum of (5.1), we need a local concavity assumption:

Assumption F3: The functions $f_{ds}(\cdot)$ are concave in y_s, whenever they are differentiable.

The linear model has $f_{ds} = a_{ds}(y_s + b_s)$ and, therefore, meets this requirement. Assumptions F2-F3, make it possible to zoom in on the structure of the site specific model. Suppose that the model function and the net precipitation also depend on site specific stocks or state variables k_s (an M-dimensional real vector of structural characteristics), and that this dependence also holds for their derivatives. The task is to price these stocks, that is to measure how much of the value of the gross output of this site can be attributed to them. Model (4.2) now extends to:

$$\begin{array}{lll} & max_{k,y}\ w(y) & \\ subject\ to & & \\ & y = F(y, b(k), k), & (p) \\ & k = \bar{k}, & (y) \end{array} \tag{5.2}$$

where we require:

Assumption K1: The functions w and F are continuous in (y,b,k) whenever they are differentiable in y.

Now to attribute the value at site s, we only need to consider the site-specific problem:

Proposition 5 (valuation of stocks and characteristics) If assumptions F1-F3, K1, W hold, then, after running procedures (4.2), and (4.10), the site specific problems

$$\begin{array}{ll} P_s(p, \bar{k}_s) = max_{k_s, y_s}\ w'_s y_s + \sum_{d \in N_s^d} p_d f_{ds}(y_s, b_s(k_s), k_s) - p_s y_s & \\ subject\ to & \\ \qquad k_s = \bar{k}_s & (y_s), \end{array} \tag{5.3a}$$

computes the Lagrange multipliers of program (5.2). Moreover,

$$y_s = \frac{\partial P_s(p, \bar{k}_s)}{\partial \bar{k}_s} = \sum_{d \in N_s^d} p_d \, \frac{\partial f_{ds}(y_s, b_s(k_s), k_s)}{\partial k_s} \tag{5.3b}$$

Proof. Once the accruals y_s and their prices p_s have been calculated, program (5.3) is well defined, all its primal values are given, and, by assumption F4, define a local optimum and the Lagrange multipliers are readily computed. Assertion (5.3b) follows from the envelope theorem, noting that application of the chain rule to the objective of (5.3a) leads to:

$$\frac{\partial P_s(p, \bar{k}_s)}{\partial \bar{k}_s} = \sum_{d \in N_s^d} p_d \, \frac{\partial f_{ds}(y_s, b_s(k_s), k_s)}{\partial k_s} + \sum_{d \in N_s^d} p_d \, \frac{\partial f_{ds}(y_s, b_s(k_s), k_s)}{\partial y_s} \frac{\partial y_s}{\partial k_s}$$

where by assumption F2 is differentiable almost everywhere, and evaluation is done at a differentiable point. The second term drops out, since $\frac{\partial y_s}{\partial k_s} = 0$ in the optimum. ■

We note that this nonlinear, and possibly non-convex program merely serves as a device to calculate the Lagrange multiplier y_s because the optimum is known, even highly non-convex optimization problems can be addressed in this way, at low computational costs (e.g. in GAMS, see Brooke et al. 1998). Alternatively, (5.3b) makes it possible to compute the multiplier directly. This proposition makes it possible to construct, in addition to the budget specified in section 3, a full economic account of the hydrological model f under study at this site. Specifically, the *Delivery Account* is composed as:

	Value in end use:	$w_s' y_s$
+	Value of delivery:	$\sum_{d \in N_s^d} p_d \, f_{ds}(y_s, b_s(k_s), k_s)$
=		
	Cost of accrual:	$p_s y_s$
+	Net profit:	$\mathbf{P}_s$,

for every site s. Hence, at sites without end user the profit will be equal to the value of delivery (output) minus the cost of accrual (input). Since on aggregate the cost of accrual balances with the value of delivery, summation over sites s leads, after summation over all sites, to the identity:

	Total value in end use:	$\sum_s w_s' y_s$
=		
	Total profit:	$\sum_s \mathbf{P}_s$,

A delivery account becomes more informative if it is supplemented by a profit account that can attribute the net profit to well specified factors k. To construct a profit account, impose the following assumption, without any loss of generality:

Assumption F4: the functions $f_{ds}(y_s, b_s(k_s), k_s)$, are homogeneous of degree one in (y_s, k_s).

Proposition 6 (exhaustion of value) If assumption F4 holds, in addition to assumptions F1-F3, K1, W of proposition 5, hold, then the characteristics prices y_s exhaust all value.

Proof. There is no loss of generality since for any function $\hat{f}_{ds}(y_s, \hat{k}_s)$ that is not homogeneous of degree one in $(y_s, \hat{k}_s)$, we can define an extended vector of characteristics $k_s = (\hat{k}_s, n_s)$, with fixed factor $n_s = 1$, to obtain an extended function $f_{ds} = n_s \hat{f}_{ds}(y_s / n_s, \hat{k}_s / n_s)$, that possesses this homogeneity property. Then, under Assumption F4, we have full value exhaustion, and can therefore attribute the (possibly negative) net profit to characteristics, that may also have negative values.■

The resulting *Profit Account* has the form:

	Net profit:	P_s
=		
	Value of characteristic 1:	$y_{1,s} \bar{k}_{1,s}$
	*	*
	*	*
	*	*
+	Value of characteristic M:	$y_{M,s} \bar{k}_{M,s}$

This value decomposition makes it possible to assess the relevance of the contribution of any specific model coefficient or component. Also, since the sum of net profits over all sites balances with the total value of end use, we have:

	Total value in end use:	$\sum_s w_s' y_s$
=		
	Total profit:	$\sum_s \sum_{j=1}^{M} y_s \bar{k}_{j,s}$.

Several applications can be envisaged.

Delivery structure. The approach can be used for the following exercises. Suppose that we want to know where a given city gets its water from. Then, we attribute extraction value of unity to that city only, and construct the price and value map. Conversely, if we want to know the destinations of water from a given source we may introduce a marker substance and think of it as being mixed in the water, run it through the model, and evaluate the concentrations at all destinations.

Sampling and aggregate accounts. Since the user prices p_s do not depend on it, it is not required to price the structural characteristics y_s at all sites. Moreover, because all value is exhausted, it is possible to present aggregate accounts for larger geographic areas, possibly including statistics on the internal variability.

Risk and uncertainty. It is easy to run the hydrological model under various external circumstances, say, to reflect climatic variability. For example, shock r may occur with given probability P_r, and enter all the functions in the model. Then, the size of program (5.2) becomes even larger, and if expected "welfare" maximization is accepted as value criterion, it will read:

$$
\begin{aligned}
& V(\bar{k}) = max_{k, y_r, all\ r} \ \Sigma_r P_r w_r(y_r) \\
& \quad subject\ to \\
& \quad\quad y_r = F_r(y_r, b_r(k), k) \qquad\qquad (p_r) \\
& \quad\quad k = \bar{k}. \qquad\qquad\qquad\qquad (y_r)
\end{aligned}
\tag{5.4}
$$

Fortunately, we can solve the model and the valuation for every shock separately, and since the expected price of the characteristics satisfies:

$$\tilde{y} = \frac{\partial V}{\partial \bar{k}} = \Sigma_r P_r \frac{\partial V_r}{\partial \bar{k}} = \Sigma_r P_r y_r ,$$

it is straightforward to derive an aggregate account for characteristics, with entries $\tilde{y}_m \bar{k}_m$. We note that the non-linearity of the utility function may be used to reflect the risk aversion of the decision maker, and, consequently, the prices of characteristics $\tilde{y}$ will fully account for it.

Calibration. It is worthwhile to mention that the valuation is not necessarily financial. It only expressed the contribution to the objective. In particular, this objective might be a log-likelihood function, e.g. $-\frac{1}{2S}\Sigma_s(y_s - \tilde{y}_s)^2$ in case of least squares, with $\tilde{y}_s$ as the given observation of the accruals as dependent variable, at sites where observations where available, and the characteristics $\hat{k}_s$ as independent variables to represent model parameters. Valuation plays a useful role in this respect, since it maps out the marginal contribution of individual sites to the model error $w'_s = \frac{1}{S}(\tilde{y}_s - y_s)$ that acts like an amenity value f_s since it does not affect the flow itself. This makes it possible to locate the sites that most critically affect the predicted hydrograph. Similarly, if the interest is in another criterion, say, the deposition of sediments, the price map will show the marginal contributions to it. Specifically, in (5.3) we can given the prices $\boldsymbol{p}$, evaluate the marginal contribution y_s of characteristics k_s at site s. Now suppose that the first characteristic refers to a coefficient c of the hydrological model that is common to all sites. Usually, some of the contributions will be positive and other negative. Then, the mean contribution of this coefficient to the likelihood can be calculated as $\sum_s \boldsymbol{y}_{1s}$, giving the steepest ascent direction of change in this coefficient: increase it when positive and decrease when negative, and a local optimum is found when it is zero. This also applies when c is a vector. We return to calibration in section 7. In the appendix we present the variance decomposition accounts into which the value accounts can be embedded.

Section 6
Dynamics and steady state

Recall from (1.4) above that hydrological models generally follow a dynamic approach that serves the double purpose of maintaining approximate consistency and describing the actual system. So far, we focused on consistency, and found that it can be achieved without iterative approximation. The hydrological model was taken to apply to a single period, of indefinite but finite length, and possibly consisting of a large number of consecutive time intervals, with each site in period t fully situated downstream of all those in period $t-1$. This has the important advantage of keeping the procedures for model solution completely separate from the model specification itself, and made it possible to address the problem of valuation. We are now ready to make time more explicit and to exploit the sparseness of links between periods. This will enable us to deal with the steady state of a model with an infinite time horizon. For this, we follow the approach which is common in capital theory (Morishima, 1973, Stokey and Lucas, 1989), interpret the models of the previous section as applying to a single hydrological cycle (a year), and consider a sequence of years indexed $t = 0,1,...$.[8]

To highlight the main dynamic aspects we start with the formulation of a linear dynamic model, and prove that it has a unique steady state, i.e a situation that can replicate itself indefinitely. We distinguish between availability q_s, accrual to lower sites y_s, both scalars, and the nonnegative vector of stocks k_s, that are in this simple model taken to represent a carryover stock of water, remain constant in a steady state.

$$\begin{aligned} q_s^t &= y_s^t + k_s^t + b_s \\ y_s^t &= \sum_{u \in N_u^d} \boldsymbol{a}_{su} q_u^t \\ k_s^{t+1} &= m_s q_s^t , \end{aligned} \tag{6.1}$$

where the accumulation fraction m_s is non-negative and less than unity. In vector notation covering all sites, this can be written as:

$$\begin{aligned} y^t &= Aq^t \\ q^t &= y^t + k^t + b \qquad , t = 0,1, \dots \\ k^{t+1} &= Mq^t , \end{aligned} \tag{6.2}$$

given $k^0 = k_o$.

[8] A discrete time framework can more easily accommodate the ranking procedure of earlier propositions.

Proposition 7 (existence, uniqueness and stability of steady state of linear model). If assumptions A-B hold, and the accumulation fractions m_s are less than unity, then model (6.2) is well defined, has a unique steady state, and converges to it.

Proof. The first two equations in (6.2) can be used to solve for q^t and permit to write $q^t = (I - A)^{-1}(k^t + b)$, where $(I - A)^{-1}$ is nonsingular and non-negative, since every row consists of A consists of nonnegative fractions whose sum is less than unity. Substitution into the stock equation of (6.2) gives the stock dynamics:

$$k^{t+1} = M(I - A)^{-1}(k^t + b), t = 0,1, \ldots \tag{6.3}$$

given $k^0 = k_o$, and nonnegativity of stocks is ensured, for any nonnegative initial stock k_o. Therefore, the model is well defined. Its dynamics are determined by the transition matrix $M(I - A)^{-1}$, whose eigenvalues are equal to the largest element among the product of the accumulation factor and the diagonal elements of the second term, which are all equal to unity (Noble, 1969 p. 316), while all m_s are less than unity. Hence, the iteration converges to the same steady state, that does not depend on k_o and is, therefore, unique. ■

Next, we generalize the formulation, and formulate nonlinear stock (characteristics) adjustment equations in which the end-of-period stock depends on the initial stock and the accrual at this site:

$$\tilde{k}_s = Q_s(y_s, k_s), \tag{6.4}$$

where k_s refers to a beginning of period stock and $\tilde{k}_s$ to an end-of-period stock. These adjustment equations can be appended to the, period-specific, nonlinear flow model and lead to:

$$\begin{aligned} k_{t+1} &= Q(y_t, k_t) \\ y_t &= F(y_t, b(k_t), k_t), \end{aligned} \tag{6.5}$$

where the end-of-period stock of period t is the beginning-of-period stock of $t+1$: $k_{s,t+1} = \tilde{k}_{s,t}$, starting from k_o. For every time period, we can use Proposition 1 to solve for y_t, exactly, and adjust the stocks accordingly.

Second, turning to valuation, the classical difficulty to be addressed is that the value of flows p_t depends on the future (downstream) use, and that the future is of indefinite length. Hence, we must introduce an end-of-period valuation y_{t+1} on stocks and a discount factor b. The period specific problem is written (as in Stokey and Lucas, 1989):

$$\begin{aligned}
&max_{k_t, k_{t+1}, y_t}\ w(y_t) + b y_{t+1}^T k_{t+1} \\
&\text{subject to} \\
&k_{t+1} = Q(y_t, k_t) \qquad (l_t) \\
&y_t = F(y_t, b(k_t), k_t) \qquad (p_t) \\
&k_t = \bar{k}_t, \qquad (y_t)
\end{aligned} \tag{6.6}$$

for given end-of-period value of stocks y_{t+1}. The problem is obviously that y_{t+1} is unknown. Therefore, we focus on valuation at a steady state, where $k_{t+1} = k_t$, and every period is a perfect replication of the previous one. This is not necessarily realistic, but it permits to identify price and tax structures that can support environmental conservation. We make an assumption that reflects the requirement of limited accumulation ($m_s < 1$) of Proposition 7, and is commonly met in hydrology models.

Assumption K2: The set K_s is compact and for every site, the stock adjustment equation $K_s \times R_+ \to K_s, Q_s$ is continuous, and homogeneous of degree one; the dynamic iteration (6.1) converges to such a steady state, and this steady state is unique.

For simplicity we have formulate the convergence requirement in a non-constructive form, but for a given model the property is easily implemented, since it essentially demands that part of stock increase runs off to lower areas, and that policies are in place that make it technically possible to stop environmental degradation.

In the steady state, the end-value y_{t+1} and should coincide with y_t. If the model was small, it would be possible to conduct an iteration over this value, until convergence but this is most impractical given the extremely large size of this vector of $C \times M$. To overcome this difficulty, we slightly modify the program, and replace k_t in the accrual equation by k_{t+1}, which obviously does not affect the primal values, since both are equal. Consequently, denoting the initial, steady state stock by k^*, we can rewrite (6.6).

$$\begin{aligned}
&max_{k, \tilde{k}, y}\ w(y) + b\bar{y}^T \tilde{k} \\
&\text{subject to} \\
&\tilde{k} = Q(y, k) \qquad (l) \\
&y = F(y + b(\tilde{k}), \tilde{k}) \qquad (p) \\
&k = k^*, \qquad (y)
\end{aligned} \tag{6.7}$$

Now the main step is that for this modified program, we can solve for the equilibrium stock price directly as a Lagrange multiplier of the program, while dropping the stock valuation in the objective.

Proposition 8 (valuation in the steady state) If assumptions F1-F4, K1, K2 and W hold, then, after running procedures (4.2), and (4.10), the prices of characteristics can be derived as Lagrange multipliers of:

$$
\begin{aligned}
P_s(p,\hat{k}_s) = max_{\tilde{k}_s, y_s}\; & w_s' y_s + \sum_{d \in N_s^d} p_d f_{ds}(y_s + b_s(\tilde{k}_s), \tilde{k}_s) - p_s y_s \\
& subject\ to \\
& \tilde{k}_s = Q_s(b\tilde{k}_s + (1-b)k_s, y_s) \\
& (1-b)k_s = \hat{k}_s, \qquad\qquad (y_s)
\end{aligned}
$$

(6.8)

for given $\hat{k}_s = (1-b)k_s^*$.

Proof. We first show that the multiplier y of (6.7) can be computed from a program without unknown prices in the objective:

$$
\begin{aligned}
& max_{k,\tilde{k},y}\; w(y) \\
& subject\ to \\
& \tilde{k} = Q(y, b\tilde{k} + (1-b)k) \qquad (l) \\
& y = F(y + b(\tilde{k}), \tilde{k}) \qquad (p) \\
& (1-b)k = \hat{k}, \qquad (y)
\end{aligned}
\tag{6.9}
$$

for $\hat{k} = (1-b)k^*$. To verify the equivalence, we compare the first-order conditions. For (6.7) these are:

$$
\begin{aligned}
w_y'^{T} &= l^T Q_y' - p^T(I - F_y') \\
-l^T + b y^T &= p^T F_{\tilde{k}}' \\
y^T &= l^T Q_k' .
\end{aligned}
\tag{6.10}
$$

Substitution of y into the second equation yields:

$$
0 = l^T(I - bQ_k') + p^T F_{\tilde{k}}', \tag{6.11}
$$

and leads to the first-order conditions of (6.7). ■

In short, the value transmission mechanism is as follows. The stock dynamics determine the accruals in the steady state, which in turn determine the prices via their effect on marginals $\tilde{a}$ a well as their effect on end-values u'. Hence, while there is a direct impact of flow prices on stock prices, the converse does not hold, essentially because of the possibility to eliminate the stock prices from the objective in (6.7).

Finally, note that the specification of the dynamic model (6.5) and the steady state implicitly consider a full hydrological cycle of, say, one year. This neglects intermediate time-steps of months or days. Fortunately, the approach is easily adapted to calculations within the cycle. For this one runs (6.5) for the shorter time period, say, a month, where the stock variable distinguishes the physical characteristics by months, and hence, through its adjustment, generates a different precipitation for every month. Next, one looks for a steady state on an annual basis, i.e. where all years yield the same pattern over months. At this steady state one evaluates the monthly site specific prices, for every month separately. And it is only when it comes to the stock valuation of Proposition 8 that all months have to be considered simultaneously, but this is tractable since it only has to be done on a site-by-site basis.

Section 7
Optimal water control

Not surprisingly, in this spatial context, the problem of optimal water control also suffers from the curse of dimensionality, since it in principle requires iteration over the parameters values of the model. Fortunately, the linear primal and dual procedures of propositions 1 and 2 can be applied here as well, essentially because in this linear case, the valuation procedure can be run ahead of the physical flows. We describe conditions under which both the delivery fractions a_{ds} and the net precipitations b_s can be set optimally.

First, with respect to the fractions, we observe that the key requirement is that they should be independent of flows, rather than fixed. In particular, they may depend on prices, and this makes it possible to determine the optimal canalization, i.e. the optimal (downhill) routing of water. For example, suppose that t_{sd} denotes the fixed unit cost of transportation, and that delivery fractions and retention factors are constrained by nonnegative bounds such that $\sum_{d\in N_s^d} \underline{a}_{ds} + \underline{r}_s < 1$, for $\underline{a}_{ds} \le \overline{a}_{ds}$ and $\underline{r}_s \le \overline{r}_s$, which reflect the limited controllability of flows. Suppose also that $t_{ds}\underline{a}_{ds} = 0$, and that prices p_s and f_s are nonnegative. Then, these fractions and factors can be set so as to maximize profits from total delivery D_s, to solve the linear program:

$$D_s = \max_{\overline{a}_{ds} \ge a_{ds} \ge \underline{a}_{ds},\, \overline{r}_s \ge r_s \ge \underline{r}_s} \{ \textstyle\sum_{d\in N_s^d} a_{ds}(p_d - t_{ds}) + r_s p_s \mid \sum_{d\in N_s^d} a_{ds} + r_s = 1 \}, \qquad (7.1)$$

where we note that the amenity value does not enter the program because it does not depend on the decision (as it would if we allowed for optimal choices of land use at every site. Consequently, procedure (3.3) can be modified into:

$$p_s = D_s + f_s, \text{ for } s = s_\ell, \text{ and } \ell = C, C-1, \ldots, 1. \qquad (7.2)$$

The solution of (7.1) is straightforward. After ranking $(p_d - t_{ds})$, $d \in N_s^d$, and p_s in decreasing order, the shares are set in two stages. The first is to fulfil the lower bounds. The second proceeds by filling the most profitable slot, until it reaches its lower bound, and so on until the slots sum to unity. Clearly, the cost of water transportation through the canals will now affect the price relations, but we reiterate that the recursion rules out any pumping of water to more elevated grid cells.[9]

[9] Yet, it is possible to include bridging connections. For example, the pumping of water over a mountain pass, to a point lower than its source, can be represented as a costly transportation to an economically adjacent, albeit not spatially contiguous cell. The optimum will then determine whether this is an economically viable channel, and how much water would flow through it.

Second, given the unit value p of the accruals, we can optimally determine the net precipitation $b_s(k_s, p)$, that now includes the extraction by private agents (consumers or producers) and water reservoirs, on the basis of the profit maximization:

$$\begin{aligned} & max_{b_s, c_s, v_s \geq 0} \; p\, b_s + p_s c_s - v_s \\ & \text{subject to} \\ & c_s \leq g_s(v_s, k_s) \\ & b_s + c_s \leq q_s(k_s) \end{aligned} \tag{7.3}$$

where c_s is a permanent extraction, p_s the price of this extraction – there might be several alternative uses to be chosen from, say, in farming, fishing or industrial use – and v_s is the non-water input, say, the fuel for the pump, taken to have a price of unity that enters the extraction function.

Assumption G: The extraction function g_s is homogeneous of degree one in k_s, and strictly concave in v_s and the resource function q_s is homogeneous of degree one, almost everywhere on K_s, continuous, and such that $0 < g_s(0, k_s) \leq q_s(k_s)$.

Finally, this optimal extraction module can readily be included within the linear model of Proposition 1 but, for generality, we incorporate it in the steady state model of Proposition 8.

Proposition 9 (Optimal control): If assumptions A, G and K1-K2 hold, then the accruals and extractions obtained by the three-step procedure of (a) running sequence (7.2); (b) evaluating (7.3), and (c) running sequence (3.1), solves the program:

$$\begin{aligned} & max_{v_j, b_s, c_s, y_s \geq 0, z_{ns} \geq 0, k_h, \tilde{k}_h} \; \Sigma_s f_s y_s + \Sigma_s p_s c_s + \Sigma_h y_h k_h - \Sigma_j v_j - \Sigma_s \Sigma_{n \in N_s^d} t_{ns} z_{ns} \\ & \text{subject to} \\ & \Sigma_{n \in N_s^d} z_{ns} = y_s + b_s \qquad (p_s) \\ & y_n = \Sigma_{s \in N_n^u} z_{ns} \\ & c_s \leq g_s(v, k) \\ & b_s + c_s \leq q_s(k) \\ & k_h = Q_h(y, \tilde{k}) \\ & \tilde{k}_h = k_h^* \qquad (y_h) \end{aligned} \tag{7.4}$$

Proof. This is a modification of program (6.4). After running sequence (7.1)-(7.2), we can solve (7.3), which, by strict concavity of Assumption G has a unique optimum. Then, we can run (3.1), and turn to propositions 5,6,8, to derive stock valuation and evaluate the accounts. ■

Alternatively, if g_s is linear in v_s up to an upper bound: $g_s(v_s, k_s) = min(v_s / g_s, \bar{g}_s(k_s))$ for $\bar{g}_s(k_s) \leq \bar{q}_s(k_s)$, the solution of program (7.3) is not necessarily unique, but this program can be solved, nonetheless. For this, one defines and evaluates the unit profit of extraction:

$$m_s = p_s - g_s, \tag{7.5}$$

and solves (7.3) as:

$$b_s(p_s, m_s, k_s) = \{ q_s(k_s) \text{ if } p_s \geq m_s, \text{and } \bar{g}_s(k_s) \text{ otherwise} \}. \tag{7.6}$$

This formulation makes it possible to deal with the optimal regulation of reservoirs, and may lead to storage even at sites where water has zero extraction price p_s. In this case, a negative accrual value downstream, say, because of flood damage, triggers the opening of the reservoir upstream as soon as the damage exceeds the pumping costs, i.e. as $p_s < -g_s$.

However, the problem of optimal control of reservoirs typically arises in the context of the hydrological cycle. Given the monthly prices, we can determine optimal reservation for every site separately, by solving the simple linear program for all months m simultaneously:

$$\begin{aligned} & max_{b_{sm}, c_{sm} \geq 0, m=1,\ldots,12} \Sigma_m (p_{sm} b_{sm} + p_{sm} c_{sm}) \\ & subject\ to \\ & \quad c_{sm} \leq \bar{g}_{sm} \\ & \quad b_{sm} + c_{sm} \leq \bar{q}_{sm} + w_{sm} c_{s,m-1} \qquad (c_{sm}) \end{aligned} \tag{7.7}$$

where w_{sm} is a fixed carryover fraction, price p_{sm} is taken to be net of extraction cost g_{sm} and given $c_{s,0}$.

Post-optimal analysis

Though the linear framework[10] only allows for lower bounds on extraction that maintain nonnegative net precipitation, it permits, once the model has been solved, to conduct post-optimal calculations that can determined how much subsidy would be required to channel additional water to every site. For this, one evaluates for given routing cost t_{su} from u to s, the subsidies needed to bridge the price gap as:

$$s_s = min_{u \in N_s^u}(p_u - p_s + t_{su}), \qquad (7.8)$$

where the linearity also ensures that if the subsidy does not exceed s_s, it will not affect the prices p_s, and hence, the upstream flow, provided there is no (new) subsidy given upstream. The subsidy can also be looked at as an upward adjustment of the amenity price f_s. In fact, this stepwise adjustment, can be repeated in several rounds and interpreted as a way to mimic the competitive bidding for water by several end-users. We also note that, as long as the problem has the structure of a, possibly nonlinear, convex program, it can be decentralized, and the optimum can be achieved as a market equilibrium between competing, site-specific, profit maximizing agents.

Procedure (7.1)-(7.2) can also be used to identify sites where uphill pumping would be profitable. For this, one would in program (7.4) replace the directional restrictions $n \in N_s^d$ and $n \in N_s^u$ by the neighborhood restriction $n \in N_s$. Consequently, the matrix of the problem loses its recursive (triangular) form, and in our case with a 1000×1000 grid and contains, for rectangular cells as much as $2 \cdot 8 \cdot G^2 = 16 \cdot 10^6$ unknown variables, which makes it less suitable for treatment by a regular transportation algorithm in, say, GAMS (Brooke et al. 1998).[11] Additional restrictions can greatly facilitate the solution procedure. Specifically, after solving (7.4), we can conduct a post-optimal analysis. Clearly, the possibility of pumping implies that the solution without it may not be globally optimal. However, since this program is convex, it is possible identify by post-optimal calculations, the sites where pumping might be profitable, as follows.

---

[10] In the nonlinear model, the non-negativity requirement on net precipitation b_s can be dropped, in the sense that a distinction can be made between the target b_s and the realization $\tilde{b}_s = max(b_s, -y_s)$, that enters the specification of the nonlinear function *F*.

[11] With rectangular cells, there are eight neighbors. The comparison is not fully adequate, since the cells at the boundaries of the region have less neighbors, and, more importantly, the dual program only contains $2G$ variables, of which at most G are positive. Hence, iterations can be formulated in a G-dimensional space. Nonetheless, procedure (7.1)-(7.2) is superior, whenever applicable, since it avoids iteration altogether.

Proposition 10 (location of pumping stations): Uphill pumping is not profitable in the modified transportation problem with neighborhood restrictions only, if and only if the unit profit on transport:

$$w_{us} = p_u - t_{us} - p_s, \quad for\ all\ u \in N_s^u, \tag{7.9}$$

found in the optimum of (7.4) to any of the uphill neighbors is non-positive. Moreover, if the profits are taxed away whenever they are positive, the calculation identifies sites where pumping could be effectuated simultaneously and without losses.

Proof. The transportation component in (7.4) is a linear program, and hence convex. Therefore, any non-optimality of a proposed solution necessarily reflects in the violation of a first-order condition. Conversely, if this unit profit is non-positive for all s, the procedure has found a global optimum. Now if profits are taxed away, pumping will not affect the prices $\boldsymbol{p}$ obtained without it, and therefore, can be effectuated simultaneously, and without losses.■

In this proposition, the taxation of profit is required to avoid the upward cascading effect, whereby pumping would change the price structure altogether. Moreover, allowing for profit maximizing uphill flows would change the hydrology of the system, and undermine the recursivity assumption A. However, the quantities pumped can often be treated as exogenous, since they depend on the capacity of the installation. In this case, the profitability triggers an adjustments of net precipitation at both ends of the pipe, the recursive structure is maintained, and taxation can be dispensed of.

Optimal control in the nonlinear model

So far, we only dealt with the linear model, extended with possibly non-linear stock adjustment equations. Central was the requirement that unit values $\boldsymbol{p}$ could be determined ahead of the physical flows. In the nonlinear case, the flows are to be determined ahead of the prices, and while post-optimal analysis is straightforward, optimal control requires iteration. An obvious strategy for dealing with the non-linear case would be consider a sequence of linear approximations, in which the dual would use coefficients that differ from those of the primal. As in proposition 4, the dual would work with marginals $\tilde{\boldsymbol{a}}$, whereas the primal would work with the average outflow coefficients $\boldsymbol{a}$, evaluated on the basis of the accruals from previous iterations. However, the convergence of such a procedure is not assured. Alternatively, use can be made of the fact that Lagrange multipliers themselves actually measure the gradient of the value function.

Specifically, in (7.1) shifting a in the direction of profitability of (7.9) will raise the objective, also in the nonlinear case. Hence, in the nonlinear case as well as in the case with pumping, (7.1) can be replaced by an algorithm that changes the destination a in small steps, driven by the sign of a moving average over iterations w_{us} rather than moving in one iteration to the most profitable destination.

More generally, the nonlinear model (4.3) can be extended to include an extraction vector c, with a continuous objective $u(y,c)$ and a continuous share function $H(y,c)$. This program becomes, for a fixed value of $c = c^j$:

$$
\begin{aligned}
V(c^j) = \quad & \max_{c,y} w(y,c) \\
& \text{subject to} \\
& y = H(y,c)(y + b(c)) && (p^j) \\
& c = c^j, && (x^j)
\end{aligned}
\tag{7.10}
$$

and run the iterative process of a gradient algorithm

$$
c^{j+1} = c^j + z_j x(c^j), \qquad j = 0,1,... \tag{7.11}
$$

for step-length coefficient z_j starting at given c^0, and where $x(c^j)$ is a Lagrange multiplier and also a subgradient of V. If $x(c^j)$ is continuous, then, for a small enough step length, the procedure (7.11) will converge to a local optimum of $V(c)$ w.r.t. to c (Ortega and Rheinboldt, 1970). Ensuring convergence to a local optimum when the subgradient is non-unique, and hence discontinuous, is standard in nonsmooth optimization; and requires a suitable randomizes around c^j to compute $x(c^j)$ as a moving average over the sample (see e.g. Ermoliev and Norkin,1997). This will ensure convergence to a local optimum, which obviously is the global optimum if model (7.10) is convex in (c,y). However, calibration problems are often non-convex. Whether application of algorithms for selecting the global optimum can be effective in a non-convex problem of this size is a matter for future investigation.

Finally, we reiterate that the valuation is a technique to assess the effect of a change, and may, therefore, also serve for model calibration. Specifically, the variable c in (7.10) could refer to the coefficients to be calibrated, and the objective $w(y,c)$ to a likelihood function that penalizes deviations from empirical observations. Then, the gradient $x(c^j)$ measures the marginal contribution to the likelihood of a change in coefficient value, and iterations can be conducted to adjust these coefficients. Generally, the efficiency of the recursive procedures makes it possible to embed the flow model within a wider algorithmic setting, in which some of its parameters can be endogenized, either for water control purposes or for model calibration. This is subject for further investigation.

Section 8
Conclusion

Studying water management within a spatially explicit setting, makes it possible to describe more accurately the role of land and vegetation at watershed level, and to attribute value to the ecological processes that contribute to the supply of water, in their interaction with human activity. The duality relationship between physical flows and economic values, much in the tradition of nineteenth century, classical economic theory, including the Marxian one, further clarifies these interactions (Morishima, 1973; Sraffa, 1960). In fact, this theory applied its mechanistic theory of value to human behavior, and was duly criticized for it, but in hydrology this limitation becomes an asset, as the relationships between water particles are inherently mechanistic. In addition, classical theory, through is theory of value, suggests powerful algorithms that exploit the fundamentally recursive structure of causal relations in hydrology and maintain tractability where general purpose algorithms would succumb under the curse of dimensionality.

Once the computation of accruals proceeds from top to bottom altitudes, rather than by sequential scanning over a grid, it becomes relatively straightforward to maintain material balances between all grid cells, irrespective of the prevailing topography. Conversely, a sequence of calculations of values that moves from bottom to top makes it possible to price all the flows.

It appears that there is a basic difference between the linear model with fixed distribution of the direct downstream flows from a cell and the nonlinear one, where these are allowed to vary with the size of the flow. In the former case, it is possible to calculate the unit values of the flows at every site ahead of the calculation of flows, and, hence, to make these flows dependent on the unit values, as part of a site specific optimal water management decision, say, of reservoirs. In the latter case, the unit values depend on the flows, and the valuation has to be conducted afterwards.

The approach described also permits, once the flows and values have been determined, to zoom in on the specific scarcities at every site separately, by valuing the characteristics variables. In particular, if the system is in a steady state, it becomes in this way possible to obtain an intertemporally consistent valuation of stocks, that may also account for uncertainty.

Finally, we remark that the approach, specifically the optimization models of section 7, might have applications in other spheres of economic geography, for example, human migration or commodity trade. In migration, individuals with identical labor productivity and hence with the same homogeneity as water, move towards the city centers, where returns to labor (the inverse of altitude) are highest, unless they are employed earlier. In such a model, different wages would be paid to workers with the same skills and the labor market would operate a job lottery with frictional cost along the flow. In this context the choice of the direction of migration that determines the outflow fractions could be developed into a micro-economic decision model of the individual worker. Similarly, in a model of commodity trade the optimization model could generate the flows of agricultural commodities produced, supposing that the altitude measures the time to travel to the most profitable destination of all. Conversely, the price solution would permit to show how much every location is affected by foreign trade and trade policies, separately for different commodities. Obviously, the flow itself would cause changes in profitability, as in the non-linear version of the hydrological model, and require adjustment of marginal returns, analogous to the eventual change in altitude under erosion. Moreover, as with pumping of water, the solution found in this way may not be globally optimal. Finally, we have neglected the whole issue of property rights. Optimality was purely looked at from the efficiency perspective and disregarded all equity considerations. The owner of the land upstream may have interests that differ from those of the downstream user. While it is possible to interpret the valuation objective as a welfare criterion that reflects the agreement among stakeholders by an appropriate weighting of their respective utilities, the non-cooperative situation where different agents act strategically calls for a representation that separately maximizes from the perspective of every agent, while taking the behavior of other strategic agents as given. The hydrological model can also be used for this purpose but these are subjects for further investigation.

Appendix
Variance decomposition for calibration of the hydrological model

Section 5 presented delivery and profit accounts for economic valuation of hydrological flows y. By Assumption F4, the hydrological model was taken to be homogeneous of degree one, and hence to distribute all value $\sum_s w_s' y_s = \sum_s \sum_{j=1}^{M} y_s \bar{k}_{j,s}$ holds. Yet, this is only the monetary value at "prices" w_s'. If the value criterion $w(y)$ is homogeneous of degree on, this value is equal to it but generally this is not the case. In the context of economic valuation the difference between welfare and monetary value is referred to as the welfare surplus. By the same argument as given in the proof of Proposition 6, this surplus can be interpreted as the monetary value of some fixed factor n in $\tilde{w}(y,n) = nw(y/n)$.

However, when the criterion measures goodness of fit this interpretation is not informative. Instead, the value criterion may be interpreted as a variance that can be decomposed into the variance of observations, the variance of errors and a covariance. The covariance in turn plays the role of a monetary value. If the criterion is to minimize the sum of squared errors, we have prices $w_s' = -e_s$ at points where observations are available and $w_s' = 0$ elsewhere.

In this appendix we show how the delivery and profit accounts fit within such a decomposition of variance. For n given observations $\tilde{y}_s$ available at sites $s \in S^o$,(possibly over time), and characterisitcs k, we can run the hydrological model to obtain the flows $y_s(k)$ and compute the model error e_s. Given these values, we can conduct a variance decomposition starting from the basic identity:

$$\tilde{y}_s = y_s(k) + e_s, \qquad s \in S^o \tag{A.1}$$

We can also calculate the means over observations:

mean observed flow: $\bar{\tilde{y}} = \frac{1}{n}\sum_s \tilde{y}_s$

mean computed flow: $\bar{y} = \frac{1}{n}\sum_s y_s$

mean error: $\bar{e} = \frac{1}{n}\sum_s e_s$.

We can now turn to the decomposition of variance (all multiplied by the number of observations n), taking squares on the left and right-hand side of (A.1):

	Total variance:	$\sum_s y_s^2 - n\bar{\tilde{y}}^2$
=		
+	Variance of model:	$\sum_s y_s^2 - n\bar{\tilde{y}}^2$
+	2×Covariance:	$2\sum_s y_s \boldsymbol{e}_s$
+	Variance of error:	$\sum_s \boldsymbol{e}_s^2 - n\bar{\boldsymbol{e}}^2$
+	Bias:	$n\bar{\boldsymbol{e}}^2$.

The key step in relating this decomposition to the delivery and profit accounts is that because of homogeneity assumption F4, we can express the covariance in terms of marginal effects with respect to the characteristics k:

	Covariance	$\sum_s y_s \boldsymbol{e}_s$
=	Marginal effect	$\sum_s e_s \sum_j \dfrac{\partial y_s(k)}{\partial k_j} k_j$.

Recalling that $\dfrac{\partial w}{\partial y_s} = -e_s$ for a value criterion $w(y)$ that is the sum of squared errors at valuation points, it appears that the delivery and profit accounts of section 5 distributes the full value of the covariance at observed sites (the value of end use), to other sites where no measurement is made. Specifically, at every site we have:

	Value in end use:	$-e_s y_s$ if $s \in S^o$ and 0 otherwise
+	Value of delivery:	$\sum_{d \in N_s^d} p_d f_{ds}(y_s, b_s(k_s), k_s)$
=		
	Cost of accrual:	$p_s y_s$
+	Net profit:	$\boldsymbol{P}_s$,

and the sum of net profits at all sites balances with the negative of the covariance:

$$\sum_{s \in S^o} (-\boldsymbol{e}_s) y_s = \sum_{s \in S} \boldsymbol{P}_s \ . \tag{A.2}$$

At a stationary point of the calibration process (7.11), the contribution to the covariance will be eliminated for the coefficients that are being calibrated. This implies that their error weighted contribution

$$\sum_s e_s \frac{\partial y_s(k)}{\partial k_j} = 0 \ , \tag{A.3}$$

vanishes at sites where observations are available. Finally, we note that if the errors are correlated across observations, the criterion will account for this and the covariance terms will include variation across *s*.

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Centre for World Food Studies
SOW-VU
De Boelelaan 1105
1081 HV Amsterdam
The Netherlands

Telephone (31) 20 - 44 49321
Telefax (31) 20 - 44 49325
Email pm@sow.econ.vu.nl
www http://www.sow.econ.vu.nl/